Decay rates for the damped wave equation on the torus
With an appendix by Stéphane Nonnenmacher[*]

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Abstract

We address the decay rates of the energy for the damped wave equation when the damping coefficient b does not satisfy the Geometric Control Condition (GCC). First, we give a link with the controllability of the associated Schrödinger equation. We prove in an abstract setting that the observability of the Schrödinger group implies that the semigroup associated to the damped wave equation decays at rate $1/\sqrt{t}$ (which is a stronger rate than the general logarithmic one predicted by the Lebeau Theorem).

Second, we focus on the 2-dimensional torus. We prove that the best decay one can expect is $1/t$, as soon as the damping region does not satisfy GCC. Conversely, for smooth damping coefficients b, we show that the semigroup decays at rate $1/t^{1-\varepsilon}$, for all $\varepsilon > 0$. The proof relies on a second microlocalization around trapped directions, and resolvent estimates.

In the case where the damping coefficient is a characteristic function of a strip (hence discontinuous), Stéphane Nonnenmacher computes in an appendix part of the spectrum of the associated damped wave operator, proving that the semigroup cannot decay faster than $1/t^{2/3}$. In particular, our study shows that the decay rate highly depends on the way b vanishes.

Keywords

Damped wave equation, polynomial decay, observability, Schrödinger group, torus, two-microlocal semiclassical measures, spectrum of the damped wave operator.

Contents

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Part I
The damped wave equation

1 Decay of energy: a survey of existing results

Let (M, g) be a smooth compact connected Riemannian d-dimensional manifold with or without boundary ∂M. We denote by Δ the (non-positive) Laplace-Beltrami operator on M for the metric g. Given a bounded nonnegative function, $b \in L^\infty(M)$, $b(x) \geq 0$ on M, we want to understand the asymptotic behaviour as $t \to +\infty$ of the solution u of the problem

$$\begin{cases} \partial_t^2 u - \Delta u + b(x)\partial_t u = 0 & \text{in } \mathbb{R}^+ \times M, \\ u = 0 & \text{on } \mathbb{R}^+ \times \partial M \ (\text{if } \partial M \neq \emptyset), \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } M. \end{cases} \tag{1.1}$$

The energy of a solution is defined by

$$E(u, t) = \frac{1}{2}(\|\nabla u(t)\|^2_{L^2(M)} + \|\partial_t u(t)\|^2_{L^2(M)}). \tag{1.2}$$

Multiplying (1.1) by $\partial_t u$ and integrating on M yields the following dissipation identity

$$\frac{d}{dt}E(u,t) = -\int_M b|\partial_t u|^2 dx,$$

which, as b is nonnegative, implies a decay of the energy. As soon as $b \geq C > 0$ on a nonempty open subset of M, the decay is strict and $E(u,t) \to 0$ as $t \to +\infty$. The question is then to know at which rate the energy goes to zero.

The first interesting issue concerns uniform stabilization: under which condition does there exist a function $F(t)$, $F(t) \to 0$, such that

$$E(u,t) \leq F(t)E(u,0) \ ? \tag{1.3}$$

The answer was given by Rauch and Taylor [RT74] in the case $\partial M = \emptyset$ and by Bardos, Lebeau and Rauch [BLR92] in the general case (see also [BG97] for the necessity of this condition): assuming that $b \in \mathscr{C}^0(\overline{M})$, uniform stabilisation occurs if and only if the set $\{b > 0\}$ satisfies the Geometric Control Condition (GCC). Recall that a set $\omega \subset M$ is said to satisfy GCC if there exists $L_0 > 0$ such that every geodesic γ (resp. generalised geodesic in the case $\partial M \neq \emptyset$) of M with length larger than L_0 satisfies $\gamma \cap \omega \neq \emptyset$. Under this condition, one can take $F(t) = Ce^{-\kappa t}$ (for some constants $C, \kappa > 0$) in (1.3), and the energy decays exponentially. Finally, Lebeau gives in [Leb96] the explicit (and optimal) value of the best decay rate κ in terms of the spectral abscissa of the generator of the semigroup and the mean value of the function b along the rays of geometrical optics.

In the case where $\{b > 0\}$ does not satisfy GCC, i.e. in the presence of "trapped rays" that do not meet $\{b > 0\}$, what can be said about the decay rate of the energy? As soon as $b \geq C > 0$ on a nonempty open subset of M, Lebeau shows in [Leb96] that the energy (of smoother initial data) goes at least logarithmically to zero (see also [Bur98]):

$$E(u,t) \leq C\big(f(t)\big)^2 \left(\|u_0\|^2_{H^2(M)\cap H^1_0(M)} + \|u_1\|^2_{H^1_0(M)}\right), \qquad \text{for all } t > 0, \tag{1.4}$$

with $f(t) = \frac{1}{\log(2+t)}$ (where $H^2(M) \cap H^1_0(M)$ and $H^1_0(M)$ have to be replaced by $H^2(M)$ and $H^1(M)$ respectively if $\partial M = \emptyset$). Note that here, $\big(f(t)\big)^2$ characterizes the decay of the energy, whereas $f(t)$ is that of the associated semigroup. Moreover, the author constructed a series of explicit examples of geometries for which this rate is optimal, including for instance the case where $M = \mathbb{S}^2$ is the two-dimensional sphere and $\{b > 0\} \cap N_\varepsilon = \emptyset$, where N_ε is a neighbourhood of an equator of $\mathbb{S}^2$. This result is generalised in [LR97] for a wave equation damped on a (small) part of the boundary. In this paper, the authors also make the following comment about the result they obtain:

"Notons toutefois qu'une étude plus approfondie de la localisation spectrale et des taux de décroissance de l'énergie pour des données régulières doit faire intervenir la dynamique globale du flot géodésique généralisé sur M. Les théorèmes [LR97, Théorème 1] et [LR97, Théorème 2] ne fournissent donc que les bornes *a priori* qu'on peut obtenir sans aucune hypothèse sur la dynamique, en n'utilisant que les inégalités de Carleman qui traduisent "l'effet tunnel"."

In all examples where the optimal decay rate is logarithmic, the trapped ray is a stable trajectory from the point of view of the dynamics of the geodesic flow. This means basically that an important amount of the energy can stay concentrated, for a long time, in a neighbourhood of the trapped ray, i.e. away from the damping region.

If the trapped trajectories are less stable, or unstable, one can expect to obtain an intermediate decay rate, between exponential and logarithmic. We shall say that the energy decays at rate $f(t)$ if (1.4) is satisfied (more generally, see Definition 2.2 below in the abstract setting). This problem has already been adressed and, in some particular geometries, several different behaviours have been exhibited. Two main directions have been investigated.

On the one hand, Liu and Rao considered in [LR05] the case where M is a square and the set $\{b > 0\}$ contains a vertical strip. In this situation, the trapped trajectories consist in a family of parallel vertical geodesics; these are unstable, in the sense that nearby geodesics diverge at a linear rate. They proved that the energy decays at rate $\left(\frac{\log(t)}{t}\right)^{\frac{1}{2}}$ (i.e., that (1.4) is satisfied with $f(t) = \left(\frac{\log(t)}{t}\right)^{\frac{1}{2}}$). This was extended by Burq and Hitrik [BH07] (see also [Nis09]) to the case of partially rectangular two-dimensional domains, if the set $\{b > 0\}$ contains a neighbourhood of the non-rectangular part. In [Phu07], Phung proved a decay at rate $t^{-\delta}$ for some (unprecised) $\delta > 0$ in a three-dimensional domain having two parallel faces. In all these situations, the only obstruction to GCC is due to a "cylinder of periodic orbits". The geometry is flat and the unstabilities of the geodesic flow around the trapped rays are relatively weak (geodesics diverge at a linear rate).

In [BH07], the authors argue that the optimal decay in their geometry should be of the form $\frac{1}{t^{1-\varepsilon}}$, for all $\varepsilon > 0$. They provide conditions on the damping coefficient $b(x)$ under which one can obtain such decay rates, and wonder whether this is true in general. Our main theorem (see Theorem 2.6 below) extends these results to more general damping functions b on the two-dimensional torus.

On the other hand, Christianson [Chr10] proved that the energy decays at rate $e^{-C\sqrt{t}}$ for some $C > 0$, in the case where the trapped set is a hyperbolic closed geodesic. Schenck [Sch11] proved an energy decay at rate e^{-Ct} on manifolds with negative sectional curvature, if the trapped set is "small enough" in terms of topological pressure (for instance, a small neighbourhood of a closed geodesic), and if the damping is "large enough" (that is, starting from a damping function b, βb will work for any $\beta > 0$ sufficiently large). In these two papers, the geodesic flow near the trapped set enjoys strong instability properties: the flow on the trapped set is uniformly hyperbolic, in particular all trajectories are exponentially unstable.

These cases confirm the idea that the decay rates of the energy strongly depends on the stability of trapped trajectories.

One may now want to compare these geometric situations to situations where the Schrödinger group is observable (or, equivalently, controllable), i.e. for which there exist $C > 0$ and $T > 0$ such that, for all $u_0 \in L^2(M)$, we have

$$\|u_0\|^2_{L^2(M)} \leq C \int_0^T \|\sqrt{b}\, e^{-it\Delta} u_0\|^2_{L^2(M)} dt. \tag{1.5}$$

The conditions under which this property holds are also known to be related to stability of the geodesic flow. In particular, the works [BLR92], [LR05], [BH07, Nis09] and [Chr10, Sch11] can be seen as counterparts for damped wave equations of the articles [Leb92], [Har89a, Jaf90], [BZ04] and [AR10], respectively, in the context of observation of the Schrödinger group.

Our main results are twofold. First, we clarify (in an abstract setting) the link between the observability (or the controllability) of the Schrödinger equation and polynomial decay for the damped wave equation. This follows the spirit of [Har89b], [Mil05], exploring the links between the different equations and their control properties (e.g. observability, controllability, stabilization...). More precisely, we prove that the controllability of the Schrödinger equation implies a polynomial decay at rate $\frac{1}{\sqrt{t}}$ for the damped wave equation (Theorem 2.3).

Second, we study precisely the damped wave equation on the flat torus $\mathbb{T}^2$ in case GCC fails. We give the following *a priori* lower bound on the decay rate, revisiting the argument of [BH07]: (1.1) is not stable at a better rate than $\frac{1}{t}$, provided that GCC is not satisfied. In this situation, the Schrödinger group is known to be controllable (see [Jaf90], [Kom92] and the more recent works [AM11] and [BZ11]). Thus, one cannot hope to have a decay better than polynomial in our previous result, i.e. under the mere assumption that the Schrödinger flow is observable.

The remainder of the paper is devoted to studying the gap between the *a priori* lower and upper bounds given respectively by $\frac{1}{t}$ and $\frac{1}{\sqrt{t}}$ on flat tori. For *smooth* nonvanishing damping coefficient $b(x)$, we prove that the energy decays at rate $\frac{1}{t^{1-\varepsilon}}$ for all $\varepsilon > 0$. This result holds without making any dynamical assumption on the damping coefficient, but only on the order of vanishing of b. It generalises a result of [BH07], which holds in the case where b is invariant in one direction. Our analysis is, again, inspired by the recent microlocal approach proposed in [AM11] and [BZ11] for the observability of the Schrödinger group. More precisely, we follow here several ideas and tools introduced in [Mac10] and [AM11].

In the situation where b is a characteristic function of a vertical strip of the torus (hence discontinuous), Stéphane Nonnenmacher proves in Appendix B that the decay rate cannot be faster than $\frac{1}{t^{2/3}}$. This is done by explicitly computing the high frequency eigenvalues of the damped wave operator which are closest to the imaginary axis (see for instance the figures in [AL03, AL12]). The fact that the decay rate $1/t$ is not achieved in this situation was observed in the numerical computations presented in [AL12].

In contrast to the control problem for the Schödinger equation, this result shows that the stabilization of the wave equation is not only sensitive to the global properties of the geodesic flow, but also to the rate at which the damping function vanishes.

2 Main results of the paper

Our first result can be stated in a general abstract setting that we now introduce. We come back to the case of the torus afterwards.

2.1 The damped wave equation in an abstract setting

Let H and Y be two Hilbert spaces (resp. the state space and the observation/control space) with norms $\|\cdot\|_H$ and $\|\cdot\|_Y$, and associated inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Y$.

We denote by $A : D(A) \subset H \to H$ a *nonnegative* selfadjoint operator with compact resolvent, and $B \in \mathcal{L}(Y;H)$ a control operator. We recall that $B^* \in \mathcal{L}(H;Y)$ is defined by $(B^*h, y)_Y = (h, By)_H$ for all $h \in H$ and $y \in Y$.

Definition 2.1. We say that the system

$$\partial_t u + iAu = 0, \quad y = B^* u, \tag{2.1}$$

is observable in time T if there exists a constant $K_T > 0$ such that, for all solution of (2.1), we have

$$\|u(0)\|_H^2 \leq K_T \int_0^T \|y(t)\|_Y^2 dt.$$

We recall that the observability of (2.1) in time T is equivalent to the exact controllability in time T of the adjoint problem

$$\partial_t u + iAu = Bf, \quad u(0) = u_0, \tag{2.2}$$

(see for instance [Leb92] or [RTTT05]). More precisely, given $T > 0$, the exact controllability in time T is the ability of finding for any $u_0, u_1 \in H$ a control function $f \in L^2(0,T;Y)$ so that the solution of (2.2) satisfies $u(T) = u_1$.

We equip $\mathcal{H} = D(A^{\frac{1}{2}}) \times H$ with the graph norm

$$\|(u_0,u_1)\|_{\mathcal{H}}^2 = \|(A + \mathrm{Id})^{\frac{1}{2}} u_0\|_H^2 + \|u_1\|_H^2,$$

and define the seminorm

$$|(u_0,u_1)|_{\mathcal{H}}^2 = \|A^{\frac{1}{2}} u_0\|_H^2 + \|u_1\|_H^2.$$

Of course, if A is coercive on H, $|\cdot|_{\mathcal{H}}$ is a norm on $\mathcal{H}$ equivalent to $\|\cdot\|_{\mathcal{H}}$.

We also introduce in this abstract setting the damped wave equation on the space $\mathcal{H}$,

$$\begin{cases} \partial_t^2 u + Au + BB^*\partial_t u = 0, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}, \end{cases} \tag{2.3}$$

which can be recast on $\mathcal{H}$ as a first order system

$$\begin{cases} \partial_t U = \mathcal{A} U, \\ U|_{t=0} = {}^t(u_0, u_1), \end{cases} \quad U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ -A & -BB^* \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A^{\frac{1}{2}}). \quad (2.4)$$

The compact injections $D(A) \hookrightarrow D(A^{\frac{1}{2}}) \hookrightarrow H$ imply that $D(\mathcal{A}) \hookrightarrow \mathcal{H}$ compactly, and that the operator $\mathcal{A}$ has a compact resolvent.

We define the energy of solutions of (2.3) by

$$E(u,t) = \frac{1}{2}\big(\|A^{\frac{1}{2}}u\|_H^2 + \|\partial_t u\|_H^2\big) = \frac{1}{2}|(u, \partial_t u)|_{\mathcal{H}^2}^2.$$

Definition 2.2. Let f be a function such that $f(t) \to 0$ when $t \to +\infty$. We say that System (2.3) is stable at rate $f(t)$ if there exists a constant $C > 0$ such that for all $(u_0, u_1) \in D(\mathcal{A})$, we have

$$E(u,t)^{\frac{1}{2}} \leq C f(t) |\mathcal{A}(u_0, u_1)|_{\mathcal{H}}, \quad \text{for all } t > 0.$$

If it is the case, for all $k > 0$, there exists a constant $C_k > 0$ such that for all $(u_0, u_1) \in D(\mathcal{A}^k)$, we have (see for instance [BD08, page 767])

$$E(u,t)^{\frac{1}{2}} \leq C_k \big(f(t)\big)^k \|\mathcal{A}^k(u_0, u_1)\|_{\mathcal{H}}, \quad \text{for all } t > 0.$$

Theorem 2.3. *Suppose that there exists $T > 0$ such that System* (2.1) *is observable in time T. Then System* (2.3) *is stable at rate $\frac{1}{\sqrt{t}}$.*

Note that the gain of the $\log(t)^{\frac{1}{2}}$ with respect to [LR05, BH07] is not essential in our work. It is due to the optimal characterization of polynomially decaying semigroups obtained by Borichev and Tomilov [BT10].

This Theorem may be compared with the works (both presented in a similar abstract setting) [Har89b] by Haraux, proving that the controllability of wave-type equations in some time is equivalent to uniform stabilization of (2.3), and [Mil05] by Miller, showing that the controllability of wave-type equations in some time implies the controllability of Schrödinger-type equations in any time.

Note that the link between this abstract setting and that of Problem (1.1) is $H = Y = L^2(M)$, $A = -\Delta$ with $D(A) = H^2(M)$ if $\partial M = \emptyset$ and $H^2(M) \cap H_0^1(M)$ otherwise, B is the multiplication in $L^2(M)$ by the bounded function $\sqrt{b}$.

As a first application of Theorem 2.3 we obtain a different proof of the polynomial decay results for wave equations of [LR05] and [BH07] as consequences of the associated control results for the Schrödinger equation of [Har89a] and [BZ04] respectively.

Moreover, Theorem 2.3 provides also several new stability results for System (1.1) in particular geometric situations; namely, in all following situations, the Schrödinger group is proved to be observable, and Theorem 2.3 gives the polynomial stability at rate $\frac{1}{\sqrt{t}}$ for (1.1):

- For any nonvanishing $b(x) \geq 0$ in the 2-dimensional square (resp. torus), as a consequence of [Jaf90] (resp. [Mac10, BZ11]); for any nonvanishing $b(x) \geq 0$ in the d-dimensional rectangle (resp. d-dimensional torus) as a consequence of [Kom92] (resp. [AM11]);
- If M is the Bunimovich stadium and $b(x) > 0$ on the neighbourhood of one half disc and on one point of the opposite side, as a consequence of [BZ04];
- If M is a d-dimensional manifold of constant negative curvature and the set of trapped trajectories (as a subset of S^*M, see [AR10, Theorem 2.5] for a precise definition) has Hausdorff dimension lower than d, as a consequence of [AR10];

Moreover, Lebeau gives in [Leb96, Théorème 1 (ii)] several 2-dimensional examples for which the decay rate $\frac{1}{\log(2+t)}$ is optimal. For all these geometrical situations, Theorem 2.3 implies that the Schrödinger group is not observable.

The proof of Theorem 2.3 relies on the following characterization of polynomial decay for System (2.3). For $z \in \mathbb{C}$, we define on H the operator $P(z) = A + z^2 \operatorname{Id} + zBB^*$, with domain $D(P(z)) = D(A)$. We prove in Lemma 4.2 below that $P(is)$ is invertible for all $s \in \mathbb{R}$, $s \neq 0$.

Proposition 2.4. *Suppose that*

$$\text{for any eigenvector } \varphi \text{ of } A, \text{ we have } B^*\varphi \neq 0. \tag{2.5}$$

Then, for all $\alpha > 0$, the five following assertions are equivalent:

$$\text{The system (2.3) is stable at rate } \frac{1}{t^\alpha}, \tag{2.6}$$

$$\text{There exist } C > 0 \text{ and } s_0 \geq 0 \text{ such that for all } s \in \mathbb{R}, |s| \geq s_0,\ \|(is \operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|s|^{\frac{1}{\alpha}}, \tag{2.7}$$

$$\begin{aligned} &\text{There exist } C > 0 \text{ and } s_0 \geq 0 \text{ such that for all } z \in \mathbb{C}, \text{ satisfying } |z| \geq s_0, \\ &\text{and } |\operatorname{Re}(z)| \leq \frac{1}{C|\operatorname{Im}(z)|^{\frac{1}{\alpha}}}, \text{ we have } \|(z \operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|\operatorname{Im}(z)|^{\frac{1}{\alpha}}, \end{aligned} \tag{2.8}$$

$$\text{There exist } C > 0 \text{ and } s_0 \geq 0 \text{ such that for all } s \in \mathbb{R}, |s| \geq s_0,\ \|P(is)^{-1}\|_{\mathcal{L}(H)} \leq C|s|^{\frac{1}{\alpha}-1}, \tag{2.9}$$

$$\begin{aligned} &\text{There exists } C > 0 \text{ and } s_0 \geq 0 \text{ such that for all } s \in \mathbb{R}, |s| \geq s_0 \text{ and } u \in D(A), \\ &\|u\|_H^2 \leq C\big(|s|^{\frac{2}{\alpha}-2}\|P(is)u\|_H^2 + |s|^{\frac{1}{\alpha}}\|B^*u\|_Y^2\big). \end{aligned} \tag{2.10}$$

This proposition is proved as a consequence of the characterization of polynomial decay for general semigroups in terms of resolvent estimates given in [BT10], providing the equivalence between (2.6) and (2.7). See also [BD08] for general decay rates in Banach spaces. Note in particular that the proof of a decay rate is reduced to the proof of a resolvent estimate on the imaginary axes. By the way, this estimate implies the existence of a "spectral gap" between the spectrum of $\mathcal{A}$ and the imaginary axis, given by (2.8).

Note finally that the estimates (2.7), (2.9) and (2.10) can be equivalently restricted to $s > 0$, since $P(-is)\overline{u} = \overline{P(is)u}$.

2.2 Decay rates for the damped wave equation on the torus

The main results of this article deal with the decay rate for Problem (1.1) on the torus $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$. In this setting, as well as in the abstract setting, we shall write $P(z) = -\Delta + z^2 + zb(x)$.

First, we give an *a priori* lower bound for the decay rate of the damped wave equation, on $\mathbb{T}^2$, when GCC is "strongly violated", i.e. assuming that $\operatorname{supp}(b)$ does not satisfy GCC (instead of $\{b > 0\}$). This theorem is proved by constructing explicit *quasimodes* for the operator $P(is)$.

Theorem 2.5. *Suppose that there exists $(x_0, \xi_0) \in T^*\mathbb{T}^2$, $\xi_0 \neq 0$, such that*

$$\overline{\{b > 0\}} \cap \{x_0 + \tau\xi_0, \tau \in \mathbb{R}\} = \emptyset.$$

Then there exist two constants $C > 0$ and $\kappa_0 > 0$ such that for all $n \in \mathbb{N}$,

$$\|P(in\kappa_0)^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \geq C. \tag{2.11}$$

As a consequence of Proposition 2.4, polynomial stabilization at rate $\frac{1}{t^{1+\varepsilon}}$ for $\varepsilon > 0$ is not possible if there is a strongly trapped ray (i.e. that does not intersect $\operatorname{supp}(b)$). More precisely, in such geometry, Theorem 2.5 combined with Lemma 4.6 and [BD08, Proposition 1.3] shows that $m_1(t) \geq \frac{C}{1+t}$, for some $C > 0$ (with the notation of [BD08] where $m_1(t)$ denotes the best decay rate).

Then, the main goal of this paper is to explore the gap between the *a priori* upper bound $\frac{1}{\sqrt{t}}$ for the decay rate, given by Theorem 2.3, and the *a priori* lower bound $\frac{1}{t}$ of Theorem 2.5. Our results are twofold (somehow in two opposite directions) and concern either the case of smooth damping functions b, or the case $b = \mathbb{1}_U$, with $U \subset \mathbb{T}^2$.

2.2.1 The case of smooth damping coefficients

Our main result deals with the case of smooth damping coefficients. Without any geometric assumption, but with an additional hypothesis on the order of vanishing of the damping function b, we prove a weak converse of Theorem 2.5.

Theorem 2.6. *Let $M = \mathbb{T}^2$ with the standard flat metric. There exists $\varepsilon_0 > 0$ satisfying the following property. Suppose that b is a nonnegative nonvanishing function on $\mathbb{T}^2$ satisfying $\sqrt{b} \in \mathscr{C}^\infty(\mathbb{T}^2)$ and that there exist $\varepsilon \in (0, \varepsilon_0)$ and $C_\varepsilon > 0$ such that*

$$|\nabla b(x)| \leq C_\varepsilon b^{1-\varepsilon}(x), \quad \textit{for } x \in \mathbb{T}^2. \tag{2.12}$$

Then, there exist $C > 0$ and $s_0 \geq 0$ such that for all $s \in \mathbb{R}$, $|s| \geq s_0$,

$$\|P(is)^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \leq C|s|^\delta, \quad \textit{with } \delta = 8\varepsilon \tag{2.13}$$

As a consequence of Proposition 2.4, in this situation, the damped wave equation (1.1) *is stable at rate* $\frac{1}{t^{\frac{1}{1+\delta}}}$.

Following carefully the steps of the proof, one sees that $\varepsilon_0 = \frac{1}{76}$ works, but the proof is not optimized with respect to this parameter, and it is likely that it could be much improved.

One of the main difficulties in understanding the decay rates is that there exists no general monotonicity property of the type "$b_1(x) \leq b_2(x)$ for all $x \implies$ the decay rate associated to the damping b_2 is larger (or smaller) than the decay rate associated to the damping b_1". This makes a significant difference with observability or controllability problems of the type (1.5).

Assumption (2.12) is only a local assumption in a neighbourhood of $\partial\{b > 0\}$ (even if it is stated here globally on $\mathbb{T}^2$). Far from this set, i.e. on each compact set $\{b \geq b_0\}$ for $b_0 > 0$, the constant C_ε can be choosen uniformly, depending only on b_0, and not on ε. Hence, ε somehow quantifies the vanishing rate of the damping function b.

An interesting situation is when the smooth function b vanishes like $e^{-\frac{1}{x^\alpha}}$ in smooth local coordinates, for some $\alpha > 0$. In this case, Assumption (2.12) is satisfied for any $\varepsilon > 0$, and the associated damped wave equation (1.1) is stable at rate $\frac{1}{t^{1-\delta}}$ for any $\delta > 0$. This shows that the lower bound given by Theorem 2.5, as well as the decay rate $\frac{1}{t}$, are sharp in general. This phenomenon had already been remarked by Burq and Hitrik in [BH07] in the case where b is invariant in one direction.

Typical smooth functions not satisfying Assumption (2.12) are for instance functions vanishing like $\sin(\frac{1}{x})^2 e^{-\frac{1}{x}}$. We do not have any idea of the decay rate achieved in this case (except for the *a priori* bounds $\frac{1}{\sqrt{t}}$ and $\frac{1}{t}$).

Theorem 2.6 generalises the result of [BH07], which only holds if b is assumed to be invariant in one direction. Our proof is based on ideas and tools developped in [Mac10, AM11] and especially on two-microlocal semiclassical measures. One of the key technical points appears in Section 13: we have to construct, for each trapped direction, a cutoff function invariant in that direction and adapted to the damping coefficient b. We do not know how to adapt this technical construction to tori of higher dimension, $d > 2$; hence we do not know whether Theorem 2.6 holds in higher dimension (although we have no reason to suspect it should not hold). Only in the particular case where b is invariant in $d-1$ directions can our methods (or those of [BH07]) be applied to prove the analogue of Theorem 2.6.

Note that if GCC is satisfied, one has (on a general compact manifold M) for some $C > 1$ and all $|s| \geq s_0$ the estimate

$$\|P(is)^{-1}\|_{\mathcal{L}(L^2(M))} \leq C|s|^{-1}. \tag{2.14}$$

instead of (2.13). Estimate (2.14) is in turn equivalent to uniform stabilization (see [Hua85] together with Lemma 4.6 below).

Remark 2.7. As a consequence of Theorem 2.6 on the torus, we can deduce that the decay rate $t^{-\frac{1}{1+\delta}}$ also holds for Equation (1.1) if $M = (0,\pi)^2$ is the square, one takes with Dirichlet or Neumann boundary conditions, and the damping function b is smooth, vanishes near ∂M and satisfies Assumption (2.12). First, we extend the function b as an even (with respect to both variables) smooth function on the larger square $(-\pi,\pi)^2$, and using the injection $\imath : (-\pi,\pi)^2 \to \mathbb{T}^2$, as a smooth function on $\mathbb{T}^2$, still satisfying (2.12). Moreover, $D(\Delta_D)$ (resp. $D(\Delta_N)$) on $(0,\pi)^2$ can be identified as the closed subspace of odd (resp. even) functions of $D(\Delta_D)$ (resp. $D(\Delta_N)$) on $(-\pi,\pi)^2$. Using again the injection $\imath$, it can also be identified with a closed subspace of $H^2(\mathbb{T}^2)$. The estimate

$$\|u\|_{L^2(\mathbb{T}^2)} \leq C|s|^\delta \|P(is)u\|_{L^2(\mathbb{T}^2)} \quad \text{for all } u \in H^2(\mathbb{T}^2),$$

is thus also true on the square $(0,\pi)^2$ for Dirichlet or Neumann boundary conditions. In particular, this strongly improves the results of [LR05].

The lower bound of Theorem 2.5 can be similarly extended to the case of a square with Dirichlet or Neumann boundary conditions, implying that the rate $\frac{1}{t}$ is optimal if GCC is strongly violated.

2.2.2 The case of discontinuous damping functions

Appendix B (by Stéphane Nonnenmacher) deals with the case where b is the characteristic function of a vertical strip, i.e. $b = \widetilde{B}\mathbb{1}_U$, for some $\widetilde{B} > 0$ and $U = (a,b) \times \mathbb{T} \subset \mathbb{T}^2$. Due to the invariance of b in one direction, the spectrum of the damped wave operator $\mathcal{A}$ splits into countably many "branches" of eigenvalues. This structure of the spectrum is illustrated in the numerics of [AL03, AL12].

The branch closest to the imaginary axis is explicitly computed, it contains a sequence of eigenvalues $(z_i)_{i\in\mathbb{N}}$ such that $\operatorname{Im} z_i \to \infty$ and $|\operatorname{Re} z_i| \leq \frac{C_0}{(\operatorname{Im} z_i)^{3/2}}$. This result is in agreement with the numerical tests given in [AL12].

As a consequence, for any $\varepsilon > 0$ and $C > 0$, the strip $\left\{|\operatorname{Re} z| \leq C|\operatorname{Im}(z)|^{-3/2+\varepsilon}\right\}$ contains infinitely many poles of the resolvent $(z \operatorname{Id} - \mathcal{A})^{-1}$, so item (2.8) in Proposition 2.4 implies the following obstruction to the stability of this damped system :

Corollary 2.8. *For any $\varepsilon > 0$, the damped wave equation* (1.1) *on $\mathbb{T}^2$ with the damping function* (B.1) *cannot be stable at the rate $\frac{1}{t^{2/3+\varepsilon}}$.*

The same result holds on the square with Dirichlet or Neumann boundary conditions.

More precisely, in this situation, Lemma 4.6 and [BD08, Proposition 1.3] yield that $m_1(t) \geq \frac{C}{(1+t)^{2/3}}$, for some $C > 0$ (with the notation of [BD08] where $m_1(t)$ denotes the best decay rate).

This corollary shows in particular that the regularity conditions in Theorem 2.6 cannot be completely disposed of if one wants a stability at the rate $1/t^{1-\varepsilon}$ for small ε.

2.3 Some related open questions

The various results obtained in this article lead to several open questions.

1. In the case where b is the characteristic function of a vertical strip, our analysis shows that the best decay rate lies somewhere between $\frac{1}{t^{\frac{1}{2}}}$ and $\frac{1}{t^{\frac{2}{3}}}$, but the "true" decay rate is not yet clear.

2. It would also be interesting to investigate the spectrum and the decay rates for damping functions b invariant in one direction, but having a less singular behaviour than a characteristic function. In particular, is it possible to give a precise link between the vanishing rate of b and the decay rate?

3. In the general setting of Section 2.1 (as well as in the case of the damped wave equation on the torus), is the *a priori* upper bound $\frac{1}{t^{\frac{1}{2}}}$ for the decay rate optimal?

4. For smooth damping functions vanishing like $e^{-\frac{1}{x^\alpha}}$, Theorem 2.6 yields stability at rate $\frac{1}{t^{1-\delta}}$ for all $\delta > 0$. Is the decay rate $\frac{1}{t}$ reached in this situation? Can one find a damping function b such that the decay rate is exactly $\frac{1}{t}$?

5. The lower bound of of Theorem 2.5 is still valid in higher dimensional tori. Is there an analogue of Theorem 2.6 (i.e. for general "smooth" damping functions) for $\mathbb{T}^d$, with $d \geq 3$?

Part II
Resolvent estimates and stabilization in the abstract setting

3 Proof of Theorem 2.3 assuming Proposition 2.4

To prove Theorem 2.3, we express the observability condition as a resolvent estimate (also known as the Hautus test), as introduced by Burq and Zworski [BZ04], and further developed by Miller [Mil05] and Ramdani, Takahashi, Tenenbaum and Tucsnak [RTTT05]. For a survey of this notion, we refer to the book [TW09, Section 6.6].

In particular [Mil05, Theorem 5.1] (or [TW09, Theorem 6.6.1]) yields that System (2.1) is observable in some time $T > 0$ if and only if there exists a constant $C > 0$ such that we have

$$\|u\|_H^2 \leq C\big(\|(A - \lambda \operatorname{Id})u\|_H^2 + \|B^* u\|_Y^2\big), \quad \text{for all } \lambda \in \mathbb{R} \text{ and } u \in D(A).$$

As a first consequence, Assumption (2.5) is satisfied and Proposition 2.4 applies in this context. Moreover, we have, for all $s \in \mathbb{R}$ and $u \in D(A)$,

$$\begin{aligned} \|u\|_H^2 &\leq C\big(\|(A - s^2 \operatorname{Id} + isBB^* - isBB^*)u\|_H^2 + \|B^* u\|_Y^2\big) \\ &\leq C\big(\|P(is)u\|_H^2 + s^2\|BB^* u\|_H^2 + \|B^* u\|_Y^2\big) \end{aligned} \tag{3.1}$$

Since $B \in \mathcal{L}(Y; H)$, we obtain for $s \geq 1$ and for some $C > 0$,

$$\|u\|_H^2 \leq C\big(\|P(is)u\|_H^2 + s^2\|B^* u\|_Y^2\big) \leq C\big(s^2\|P(is)u\|_H^2 + s^2\|B^* u\|_Y^2\big).$$

Proposition 2.4 then yields the polynomial stability at rate $\frac{1}{\sqrt{t}}$ for (2.3). This concludes the proof of Theorem 2.3. $\square$

4 Proof of Proposition 2.4

Our proof strongly relies on the characterization of polynomially stable semigroups, given in [BT10, Theorem 2.4], which can be reformulated as follows.

Theorem 4.1 ([BT10], Theorem 2.4)**.** *Let $(e^{t\dot{\mathcal{A}}})_{t\geq 0}$ be a bounded $\mathscr{C}^0$-semigroup on a Hilbert space $\dot{\mathcal{H}}$, generated by $\dot{\mathcal{A}}$. Suppose that $i\mathbb{R} \cap \operatorname{Sp}(\dot{\mathcal{A}}) = \emptyset$. Then, the following conditions are equivalent:*

$$\|e^{t\dot{\mathcal{A}}}\dot{\mathcal{A}}^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} = \mathcal{O}(t^{-\alpha}), \quad \text{as } t \to +\infty, \tag{4.1}$$

$$\|(is \operatorname{Id} - \dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} = \mathcal{O}(|s|^{\frac{1}{\alpha}}), \quad \text{as } s \to \infty. \tag{4.2}$$

Let us first describe some spectral properties of the operator $\mathcal{A}$ defined in (2.4).

Lemma 4.2. *The spectrum of $\mathcal{A}$ contains only isolated eigenvalues and we have*

$$\operatorname{Sp}(\mathcal{A}) \subset \left(\big(-\frac{1}{2}\|B^*\|_{\mathcal{L}(H;Y)}^2, 0\big) + i\mathbb{R}\right) \cup \left([-\|B^*\|_{\mathcal{L}(H;Y)}^2, 0] + 0i\right),$$

with $\ker(\mathcal{A}) = \ker(A) \times \{0\}$.

Moreover, the operator $P(z)$ is an isomorphism from $D(A)$ onto H if and only if $z \notin \mathrm{Sp}(\mathcal{A})$. If this is satisfied, we have

$$(z\,\mathrm{Id} - \mathcal{A})^{-1} = \begin{pmatrix} P(z)^{-1}(BB^* + z\,\mathrm{Id}) & P(z)^{-1} \\ P(z)^{-1}(zBB^* + z^2\,\mathrm{Id}) - \mathrm{Id} & zP(z)^{-1} \end{pmatrix}. \tag{4.3}$$

The localization properties for the spectrum of $\mathcal{A}$, stated in the first part of this lemma are illustrated for instance in [AL03] or [AL12].

This Lemma leads us to introduce the spectral projector of $\mathcal{A}$ on $\ker(\mathcal{A})$, given by

$$\Pi_0 = \frac{1}{2i\pi}\int_\gamma (z\,\mathrm{Id} - \mathcal{A})^{-1} dz \in \mathcal{L}(\mathcal{H}),$$

where γ denotes a positively oriented circle centered on 0 with a radius so small that 0 is the single eigenvalue of $\mathcal{A}$ in the interior of γ. We set $\dot{\mathcal{H}} = (\mathrm{Id} - \Pi_0)\mathcal{H}$ and equip this space with the norm

$$\|(u_0, u_1)\|_{\dot{\mathcal{H}}}^2 := |(u_0, u_1)|_{\mathcal{H}}^2 = \|A^{\frac{1}{2}}u_0\|_H^2 + \|u_1\|_H^2,$$

and associated inner product. This is indeed a norm on $\dot{\mathcal{H}}$ since $\|(u_0, u_1)\|_{\dot{\mathcal{H}}} = 0$ is equivalent to $(u_0, u_1) \in \ker(A) \times \{0\} = \Pi_0\mathcal{H}$.

Besides, we set $\dot{\mathcal{A}} = \mathcal{A}|_{\dot{\mathcal{H}}}$ with domain $D(\dot{\mathcal{A}}) = D(\mathcal{A}) \cap \dot{\mathcal{H}}$. A first remark is that $\mathrm{Sp}(\dot{\mathcal{A}}) = \mathrm{Sp}(\mathcal{A}) \setminus \{0\}$, so that $\mathrm{Sp}(\dot{\mathcal{A}}) \cap i\mathbb{R} = \emptyset$.

The remainder of the proof consists in applying Theorem 4.1 to the operator $\dot{\mathcal{A}}$ in $\dot{\mathcal{H}}$. We first check the assumptions of Theorem 4.1 and describe the solutions of the evolution problem (2.4) (or equivalently (2.3)).

Lemma 4.3. *The operator $\dot{\mathcal{A}}$ generates a contraction $\mathscr{C}^0$-semigroup on $\dot{\mathcal{H}}$, denoted $(e^{t\dot{\mathcal{A}}})_{t\geq 0}$. Moreover, for all initial data $U_0 \in \mathcal{H}$, Problem* (2.4) *(or equivalently* (2.3)*) has a unique solution $U \in \mathscr{C}^0(\mathbb{R}^+; \mathcal{H})$, issued from U_0, that can be decomposed as*

$$U(t) = e^{t\dot{\mathcal{A}}}(\mathrm{Id} - \Pi_0)U_0 + \Pi_0 U_0, \quad \textit{for all } t \geq 0. \tag{4.4}$$

As a consequence, we can apply Theorem 4.1 to the semigroup generated by $\dot{\mathcal{A}}$. The proof of Proposition 2.4 will be achieved when the following lemmata are proved.

Lemma 4.4. *Conditions* (2.6) *and* (4.1) *are equivalent.*

Lemma 4.5. *Conditions* (2.9) *and* (2.10) *are equivalent. Conditions* (2.7) *and* (2.8) *are equivalent.*

Lemma 4.6. *There exist $C > 1$ and $s_0 > 0$ such that for $s \in \mathbb{R}$, $|s| \geq s_0$,*

$$\|(is\,\mathrm{Id} - \dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} - \frac{C}{|s|} \leq \|(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \|(is\,\mathrm{Id} - \dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} + \frac{C}{|s|}, \tag{4.5}$$

and

$$C^{-1}|s|\|P(is)^{-1}\|_{\mathcal{L}(H)} \leq \|(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C\left(1 + |s|\|P(is)^{-1}\|_{\mathcal{L}(H)}\right). \tag{4.6}$$

In particular this implies that (4.2)*,* (2.7) *and* (2.9) *are equivalent.*

The proof of Lemma 4.6 is more or less classical and we follow [Leb96, BH07].

Proof of Lemma 4.2. As $\mathcal{A}$ has compact resolvent, its spectrum contains only isolated eigenvalues. Suppose that $z \in \mathrm{Sp}(\mathcal{A})$, then we have, for some $(u_0, u_1) \in D(\mathcal{A}) \setminus \{0\}$,

$$\begin{cases} u_1 &= zu_0, \\ -Au_0 - BB^*u_1 &= zu_1, \end{cases}$$

and in particular

$$Au_0 + z^2u_0 + zBB^*u_0 = 0, \tag{4.7}$$

with $u_0 \in D(A) \setminus \{0\}$.

Suppose that $z \in i\mathbb{R}$, then, this yields $Au_0 - \mathrm{Im}(z)^2 u_0 + i\,\mathrm{Im}(z) BB^* u_0 = 0$. Following [Leb96], taking the inner product of this equation with u_0 yields $i\,\mathrm{Im}(z)\|B^*u_0\|_Y^2 = 0$. Hence, either $\mathrm{Im}(z) = 0$, or $B^*u_0 = 0$. In the first case, $Au_0 = 0$, i.e. $u_0 \in \ker(A)$, and $u_1 = 0$. This yields $\ker(\mathcal{A}) \subset \ker(A) \times \{0\}$ (and the other inclusion is clear). In the second case, u_0 is an eigenvector of A associated to the eigenvalue $\mathrm{Im}(z)^2$ and satisfies $B^*u_0 = 0$, which is absurd, according to Assumption (2.5). Thus, $\mathrm{Sp}(\mathcal{A}) \cap i\mathbb{R} \subset \{0\}$.

Now, for a general eigenvalue $z \in \mathbb{C}$, taking the inner product of (4.7) with u_0 yields

$$\begin{cases} (Au_0, u_0)_H + (\mathrm{Re}(z)^2 - \mathrm{Im}(z)^2)\|u_0\|_H^2 + \mathrm{Re}(z)\|B^*u_0\|_Y^2 &= 0, \\ 2\,\mathrm{Re}(z)\,\mathrm{Im}(z)\|u_0\|_H^2 + \mathrm{Im}(z)\|B^*u_0\|_Y^2 &= 0. \end{cases} \tag{4.8}$$

If $\mathrm{Im}(z) \neq 0$, then, the second equation of (4.8) together with $\mathrm{Sp}(\dot{\mathcal{A}}) \cap i\mathbb{R} \subset \{0\}$ gives

$$0 > \mathrm{Re}(z) = -\frac{1}{2}\frac{\|B^*u_0\|_Y^2}{\|u_0\|_H^2} \geq -\frac{1}{2}\|B^*\|_{\mathcal{L}(H;Y)}^2.$$

If $\mathrm{Im}(z) = 0$, then, the first equation of (4.8) together with $(\dot{A}u_0, u_0)_H \geq 0$ gives $-\mathrm{Re}(z)\|B^*u_0\|_Y^2 \geq \mathrm{Re}(z)^2\|u_0\|_H^2$, which yields

$$0 \geq \mathrm{Re}(z) \geq -\|B^*\|_{\mathcal{L}(H;Y)}^2.$$

Following [Leb96], we now give the link between $P(z)^{-1}$ and $(z\,\mathrm{Id} - \mathcal{A})^{-1}$ for $z \notin \mathrm{Sp}(\mathcal{A})$. Taking $F = (f_0, f_1) \in \mathcal{H}$, and $U = (u_0, u_1)$, we have

$$F = (z\,\mathrm{Id} - \mathcal{A})U \Longleftrightarrow \begin{cases} u_1 = zu_0 - f_0, \\ P(z)u_0 = f_1 + (BB^* + z\,\mathrm{Id})f_0. \end{cases} \tag{4.9}$$

As a consequence, we obtain that $P(z) : D(A) \to H$ is invertible if and only if $(z\,\mathrm{Id} - \mathcal{A}) : D(\mathcal{A}) \to \mathcal{H}$ is invertible, i.e. if and only if $z \notin \mathrm{Sp}(\mathcal{A})$. Moreover, for such values of z, System (4.9) is equivalent to

$$\begin{cases} u_0 = P(z)^{-1}f_1 + P(z)^{-1}(BB^* + z\,\mathrm{Id})f_0, \\ u_1 = zP(z)^{-1}f_1 + zP(z)^{-1}(BB^* + z\,\mathrm{Id})f_0 - f_0, \end{cases}$$

which can be rewritten as (4.3). This concludes the proof of Lemma 4.2. □

Proof of Lemma 4.3. Let us check that $\dot{\mathcal{A}}$ is a maximal dissipative operator on $\dot{\mathcal{H}}$ [Paz83]. First, it is dissipative since, for $U = (u_0, u_1) \in D(\dot{\mathcal{A}})$,

$$(\dot{\mathcal{A}}U, U)_{\dot{\mathcal{H}}} = (A^{\frac{1}{2}}u_1, A^{\frac{1}{2}}u_0)_H - (Au_0, u_1)_H - (BB^*u_1, u_1)_H = -\|B^*u_1\|_Y^2 \leq 0.$$

Next, the fact that $\mathcal{A} - \mathrm{Id}$ is onto is a consequence of Lemma 4.2. Hence, for all $F \in \dot{\mathcal{H}} \subset \mathcal{H}$, there exists $U \in D(\mathcal{A})$ such that $(\mathcal{A} - \mathrm{Id})U = F$. Applying $(\mathrm{Id} - \Pi_0)$ to this identity yields $(\dot{\mathcal{A}} - \mathrm{Id})(\mathrm{Id} - \Pi_0)U = F$, so that $\dot{\mathcal{A}} - \mathrm{Id} : D(\dot{\mathcal{A}}) \to \dot{\mathcal{H}}$ is onto. According to the Lumer-Phillips Theorem (see for instance [Paz83, Chapter 1, Theorem 4.3]) $\dot{\mathcal{A}}$ generates a contraction $\mathscr{C}^0$-semigroup on $\dot{\mathcal{H}}$. Then, Formula (4.4) directly comes from the linearity of Equation (2.4) (or equivalently (2.3)) together with the decomposition of the initial condition $U_0 = (I - \Pi_0)U_0 + \Pi_0 U_0$. □

Proof of Lemma 4.4. Condition (4.1) is equivalent to the existence of $C > 0$ such that for all $t > 0$, and $\dot{U}_0 \in \dot{\mathcal{H}}$, we have

$$\|e^{t\dot{\mathcal{A}}}\dot{\mathcal{A}}^{-1}\dot{U}_0\|_{\dot{\mathcal{H}}} \leq \frac{C}{t^\alpha}\|\dot{U}_0\|_{\dot{\mathcal{H}}}.$$

This can be rephrased as

$$\|e^{t\dot{\mathcal{A}}}\dot{U}_0\|_{\dot{\mathcal{H}}} \leq \frac{C}{t^\alpha}\|\dot{\mathcal{A}}\dot{U}_0\|_{\dot{\mathcal{H}}}, \tag{4.10}$$

for all $t > 0$, and $\dot{U}_0 \in D(\dot{\mathcal{A}})$. Now, take any $U_0 = (u_0, u_1) \in D(\mathcal{A})$, and associated projection $\dot{U}_0 = (\mathrm{Id} - \Pi_0)U_0 \in D(\dot{\mathcal{A}})$. According to (4.4), we have

$$E(u,t) = \frac{1}{2}\big(\|A^{\frac{1}{2}}u(t)\|_H^2 + \|\partial_t u(t)\|_H^2\big) = \frac{1}{2}|e^{t\dot{\mathcal{A}}}\dot{U}_0 + \Pi_0 U_0|_{\mathcal{H}}^2 = \frac{1}{2}\|e^{t\dot{\mathcal{A}}}\dot{U}_0\|_{\dot{\mathcal{H}}}^2,$$

and

$$|\mathcal{A}U_0|_{\mathcal{H}} = |\dot{\mathcal{A}}\dot{U}_0 + \mathcal{A}\Pi_0 U_0|_{\mathcal{H}} = \|\dot{\mathcal{A}}\dot{U}_0\|_{\dot{\mathcal{H}}}.$$

This shows that (4.10) is equivalent to (2.6), and concludes the proof of Lemma 4.4. □

Proof of Lemma 4.5. First, (2.9) clearly implies (2.10). To prove the converse, for $u \in D(\mathcal{A})$, we have

$$(P(is)u, u)_H = \big((A - s^2 \operatorname{Id})u, u\big)_H + is\|B^*u\|_Y^2.$$

Taking the imaginary part of this identity gives $s\|B^*u\|_Y^2 = \operatorname{Im}(P(is)u, u)_H$, so that, using the Young inequality, we obtain for all $\varepsilon > 0$,

$$|s|^{\frac{1}{\alpha}}\|B^*u\|_Y^2 = |s|^{\frac{1}{\alpha}-1}|\operatorname{Im}(P(is)u, u)_H| \leq \frac{|s|^{\frac{2}{\alpha}-2}}{4\varepsilon}\|P(is)u\|_H^2 + \varepsilon\|u\|_H^2.$$

Plugging this into (2.10) and taking ε sufficiently small, we obtain that for some $C > 0$ and $s_0 \geq 0$, for any $s \in \mathbb{R}, |s| \geq s_0$,

$$\|u\|_H^2 \leq C|s|^{\frac{2}{\alpha}-2}\|P(is)u\|_H^2,$$

which yields (2.9). Hence (2.9) and (2.10) are equivalent.

Second, Condition (2.8) clearly implies (2.7) and it only remains to prove the converse. For $z \in \mathbb{C}$, we write $r = \operatorname{Re}(z)$ and $s = \operatorname{Im}(z)$. We have the identity

$$((r + is)\operatorname{Id} - \mathcal{A})^{-1} = (is\operatorname{Id} - \mathcal{A})^{-1}\big(\operatorname{Id} + r(is\operatorname{Id} - \mathcal{A})^{-1}\big)^{-1}. \tag{4.11}$$

Hence, assuming

$$\|r(is\operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{1}{2}, \tag{4.12}$$

this gives

$$\left\|\big(\operatorname{Id} + r(is\operatorname{Id} - \mathcal{A})^{-1}\big)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = \left\|\sum_{k=0}^{\infty}(-1)^k\big(r(is\operatorname{Id} - \mathcal{A})^{-1}\big)^k\right\|_{\mathcal{L}(\mathcal{H})} \leq 2.$$

As a consequence of (4.11) and (2.7), we then obtain

$$\left\|((r + is)\operatorname{Id} - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leq 2\|(is\operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq 2C|s|^{\frac{1}{\alpha}},$$

for all $s \geq s_0$, under Condition (4.12). Finally, (2.7) also yields

$$\|r(is\operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq |r|C|s|^{\frac{1}{\alpha}},$$

so that Condition (4.12) is realised as soon as $|r| \leq \frac{1}{2C|s|^{\frac{1}{\alpha}}}$. This proves (2.8) and concludes the proof of Lemma 4.5. □

Proof of Lemma 4.6. To prove (4.5), we first remark that the norms $\|\cdot\|_{\dot{\mathcal{H}}}$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent on $\dot{\mathcal{H}}$, so that the norms $\|\cdot\|_{\mathcal{L}(\dot{\mathcal{H}})}$ and $\|\cdot\|_{\mathcal{L}(\mathcal{H})}$ are equivalent on $\mathcal{L}(\dot{\mathcal{H}})$. Next, we have $(is\operatorname{Id} - \dot{\mathcal{A}})^{-1}(\operatorname{Id} - \Pi_0) = (is\operatorname{Id} - \mathcal{A})^{-1}(\operatorname{Id} - \Pi_0)$ and

$$\begin{aligned}\|(is\operatorname{Id} - \dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\mathcal{H})} &= \|(is\operatorname{Id} - \dot{\mathcal{A}})^{-1}(\operatorname{Id} - \Pi_0)\|_{\mathcal{L}(\mathcal{H})} = \|(is\operatorname{Id} - \mathcal{A})^{-1}(\operatorname{Id} - \Pi_0)\|_{\mathcal{L}(\mathcal{H})} \\ &\leq \|(is\operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} + \|(is\operatorname{Id} - \mathcal{A})^{-1}\Pi_0\|_{\mathcal{L}(\mathcal{H})},\end{aligned}$$

together with

$$\begin{aligned}\|(is\operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} &= \|(is\operatorname{Id} - \dot{\mathcal{A}})^{-1}(\operatorname{Id} - \Pi_0) + (is\operatorname{Id} - \mathcal{A})^{-1}\Pi_0\|_{\mathcal{L}(\mathcal{H})} \\ &\leq \|(is\operatorname{Id} - \dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\mathcal{H})} + \|(is\operatorname{Id} - \mathcal{A})^{-1}\Pi_0\|_{\mathcal{L}(\mathcal{H})}.\end{aligned}$$

Moreover, for $|s| \geq 1$, we have

$$\|(is\,\mathrm{Id} - \mathcal{A})^{-1}\Pi_0\|_{\mathcal{L}(\mathcal{H})} = \|(is)^{-1}\Pi_0\|_{\mathcal{L}(\mathcal{H})} = \frac{1}{|s|}\|\Pi_0\|_{\mathcal{L}(\mathcal{H})} = \frac{C}{|s|},$$

which concludes the proof of (4.5).

Let us now prove (4.6). For concision, we set $H_1 = D(A^{\frac{1}{2}})$ endowed with the graph norm $\|u\|_{H_1} = \|(A + \mathrm{Id})^{\frac{1}{2}}u\|_H$ and denote by $H_{-1} = D(A^{\frac{1}{2}})'$ its dual space. The operator A can be uniquely extended as an operator $\mathcal{L}(H_1; H_{-1})$, still denoted A fo simplicity. With this notation, the space H_{-1} can be equipped with the natural norm $\|u\|_{H_{-1}} = \|(A + \mathrm{Id})^{-\frac{1}{2}}u\|_H$.

As a consequence of Formula (4.3), and using the fact that $\mathrm{Sp}(\mathcal{A}) \cap i\mathbb{R} \subset \{0\}$, there exist constants $C > 1$ and $s_0 > 0$ such that for all $s \in \mathbb{R}$, $|s| \geq s_0$,

$$C^{-1}M(s) \leq \|(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq CM(s) \tag{4.13}$$

with

$$\begin{aligned} M(s) = \Big(& \|P(is)^{-1}(BB^* + is\,\mathrm{Id})\|_{\mathcal{L}(H_1)} + \|P(is)^{-1}\|_{\mathcal{L}(H;H_1)} \\ & + \|P(is)^{-1}(isBB^* - s^2\,\mathrm{Id}) - \mathrm{Id}\,\|_{\mathcal{L}(H_1;H)} + \|sP(is)^{-1}\|_{\mathcal{L}(H)}\Big) \end{aligned} \tag{4.14}$$

On the one hand, this direcly yields for $s \in \mathbb{R}$, $|s| \geq s_0$,

$$|s|\|P(is)^{-1}\|_{\mathcal{L}(H)} \leq C\|(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}.$$

This proves that (4.2) implies (2.9).

On the other hand, we have to estimate each term of (4.14). First, using $Au = P(is)u + s^2u - isBB^*u$, we have

$$\begin{aligned} \|u\|_{H_1}^2 &= \|A^{\frac{1}{2}}u\|_H^2 + \|u\|_H^2 = \left(P(is)u + s^2u - isBB^*u, u\right)_H + \|u\|_H^2 \\ &= \mathrm{Re}\left(P(is)u, u\right)_H + (s^2+1)\|u\|_H^2 \leq C\left(\|P(is)u\|_H^2 + (s^2+1)\|u\|_H^2\right) \\ &\leq C\left(1 + (s^2+1)\|P(is)^{-1}\|_{\mathcal{L}(H)}^2\right)\|P(is)u\|_H^2, \end{aligned}$$

so that

$$\|P(is)^{-1}\|_{\mathcal{L}(H;H_1)} \leq C\left(1 + (|s|+1)\|P(is)^{-1}\|_{\mathcal{L}(H)}\right). \tag{4.15}$$

Second, the same computation for $(P(is)^{-1})^* = (A - s^2\,\mathrm{Id} - isBB^*)^{-1}$ (the adjoint of $P(is)^{-1}$ in the space H) in place of $P(is)^{-1}$ leads to $(P(is)^{-1})^* \in \mathcal{L}(H; H_1)$, together with the estimate

$$\|(P(is)^{-1})^*\|_{\mathcal{L}(H;H_1)} \leq C\left(1 + (|s|+1)\|P(is)^{-1}\|_{\mathcal{L}(H)}\right).$$

By transposition, we have ${}^t(P(is)^{-1})^* \in \mathcal{L}(H_{-1}; H)$, together with the estimate

$$\|{}^t(P(is)^{-1})^*\|_{\mathcal{L}(H_{-1};H)} \leq \|(P(is)^{-1})^*\|_{\mathcal{L}(H;H_1)} \leq C\left(1 + (|s|+1)\|P(is)^{-1}\|_{\mathcal{L}(H)}\right). \tag{4.16}$$

Moreover, ${}^t(P(is)^{-1})^*$ is defined, for every $u \in H, v \in H_{-1}$, by

$$\left({}^t(P(is)^{-1})^*v, u\right)_H = \left\langle v, (P(is)^{-1})^*u\right\rangle_{H_{-1},H_1} = \Big((A + \mathrm{Id})^{-\frac{1}{2}}v, (A + \mathrm{Id})^{\frac{1}{2}}(P(is)^{-1})^*u\Big)_H.$$

In particular, taking $v \in H$ gives

$$\left({}^t(P(is)^{-1})^*v, u\right)_H = \left(P(is)^{-1}v, u\right)_H,$$

which implies that the restriction of the operator ${}^t(P(is)^{-1})^*$ to H coincides with $P(is)^{-1}$. For simplicity, we will denote $P(is)^{-1}$ for ${}^t(P(is)^{-1})^*$.

Equation (4.16) can thus be rewritten

$$\|P(is)^{-1}\|_{\mathcal{L}(H_{-1};H)} \leq C\left(1+(|s|+1)\|P(is)^{-1}\|_{\mathcal{L}(H)}\right). \tag{4.17}$$

Then, we have $P(is)^{-1}(isBB^* - s^2\,\mathrm{Id}) - \mathrm{Id} = P(is)^{-1}A$, so that

$$\begin{aligned}\|P(is)^{-1}(isBB^* - s^2\,\mathrm{Id}) - \mathrm{Id}\,\|_{\mathcal{L}(H_1;H)} &= \|P(is)^{-1}A\|_{\mathcal{L}(H_1;H)} \leq \|P(is)^{-1}\|_{\mathcal{L}(H_{-1};H)}\|A\|_{\mathcal{L}(H_1;H_{-1})} \\ &\leq \left(1+(|s|+1)\|P(is)^{-1}\|_{\mathcal{L}(H)}\right)\end{aligned} \tag{4.18}$$

Third, for $|s| \geq 1$ we write

$$P(is)^{-1}(BB^* + is\,\mathrm{Id}) = \frac{i}{s}\left(P(is)^{-1}A - \mathrm{Id}\right), \tag{4.19}$$

and it remains to estimate the term $\|P(is)^{-1}A\|_{\mathcal{L}(H_1)}$ in (4.14). For $f \in H_1$, we set $u = P(is)^{-1}Af$. We have $u \in H_1$, together with

$$(A - s^2\,\mathrm{Id} + isBB^*)u = Af.$$

Taking the real part of the inner product of this identity with u, we find

$$\|A^{\frac{1}{2}}u\|_H^2 - s^2\|u\|_H^2 = \mathrm{Re}(Af,u)_H \leq \|Af\|_{H_{-1}}\|u\|_{H_1} \leq C\|f\|_{H_1}\|u\|_{H_1},$$

as $A \in \mathcal{L}(H_1, H_{-1})$. Hence

$$\|u\|_{H_1}^2 \leq C(1+s^2)\|u\|_H^2 + C\|f\|_{H_1}^2$$

Using (4.17), this gives

$$\begin{aligned}\|u\|_{H_1}^2 &\leq C(1+s^2)\|P(is)^{-1}A\|_{\mathcal{L}(H_1;H)}^2\|f\|_{H_1}^2 + C\|f\|_{H_1}^2 \\ &\leq C(1+s^2)\|P(is)^{-1}\|_{\mathcal{L}(H_{-1};H)}^2\|f\|_{H_1}^2 + C\|f\|_{H_1}^2 \\ &\leq C(1+s^2)\left(1+(|s|+1)\|P(is)^{-1}\|_{\mathcal{L}(H)}\right)^2\|f\|_{H_1}^2,\end{aligned}$$

and finally $\|P(is)^{-1}A\|_{\mathcal{L}(H_1)} \leq C(1+|s|)\left(1+(|s|+1)\|P(is)^{-1}\|_{\mathcal{L}(H)}\right)$. Coming back to (4.19), we have, for $|s| \geq 1$,

$$\|P(is)^{-1}(BB^* + is\,\mathrm{Id})\|_{\mathcal{L}(H_1)} \leq C\left(1+|s|\|P(is)^{-1}\|_{\mathcal{L}(H)}\right). \tag{4.20}$$

Finally, combining (4.15), (4.18) and (4.20), together with (4.13)-(4.14), we obtain for $|s| \geq 1$,

$$\|(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C\left(1+|s|\|P(is)^{-1}\|_{\mathcal{L}(H)}\right).$$

This concludes the proof of Lemma 4.6.

□

Part III

Proof of Theorem 2.6: smooth damping coefficients on the torus

To prove Theorem 2.6, we shall instead prove Estimate (2.9) with $\alpha = \frac{1}{1+\delta}$ (which, according to Proposition 2.4, is equivalent to the statement of Theorem 2.6). Let us first recast (2.9) with $\alpha = \frac{1}{1+\delta}$ in the semiclassical setting : taking $h = s^{-1}$, we are left to prove that there exist $C > 1$ and $h_0 > 0$ such that for all $h \leq h_0$, for all $u \in H^2(\mathbb{T}^2)$, we have

$$\|u\|_{L^2(\mathbb{T}^2)} \leq Ch^{-\delta}\|P(i/h)u\|_{L^2(\mathbb{T}^2)} \tag{4.21}$$

We prove this inequality by contradiction, using the notion of semiclassical measures. The idea of developing such a strategy for proving energy estimates, together with the associate technology, originates from Lebeau [Leb96].

We assume that (4.21) is not satisfied, and will obtain a contradiction at the end of Section 11. Hence, for all $n \in \mathbb{N}$, there exists $0 < h_n \leq \frac{1}{n}$ and $u_n \in H^2(\mathbb{T}^2)$ such that

$$\|u_n\|_{L^2(\mathbb{T}^2)} > \frac{n}{h_n^\delta}\|P(i/h_n)u_n\|_{L^2(\mathbb{T}^2)}.$$

Setting $v_n = u_n/\|u_n\|_{L^2(\mathbb{T}^2)}$, and

$$P_b^{h_n} = -h_n^2\Delta - 1 + ih_n b(x) = h_n^2 P(i/h_n),$$

we then have, as $n \to \infty$,

$$\begin{cases} h_n \to 0^+, \\ \|v_n\|_{L^2(\mathbb{T}^2)} = 1, \\ h_n^{-2-\delta}\|P_b^{h_n} v_n\|_{L^2(\mathbb{T}^2)} \to 0. \end{cases}$$

Our goal is now to associate to the sequence (u_n, h_n) a semiclassical measure on the cotangent bundle μ on $T^*\mathbb{T}^2 = \mathbb{T}^2 \times (\mathbb{R}^2)^*$ (where $(\mathbb{R}^2)^*$ is the dual space of $\mathbb{R}^2$). To obtain a contradiction, we shall prove both that $\mu(T^*\mathbb{T}^2) = 1$, and that $\mu = 0$ on $T^*\mathbb{T}^2$.

From now on, we drop the subscript n of the sequences above, and write h in place of h_n and v_h in place of v_n. We study sequences (h, v_h) such that $h \to 0^+$ and

$$\begin{cases} \|v_h\|_{L^2(\mathbb{T}^2)} = 1 \\ \|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta}), \quad \text{as } h \to 0^+. \end{cases} \tag{4.22}$$

In particular, this last equation also yields the key information

$$(bv_h, v_h)_{L^2(\mathbb{T}^2)} = h^{-1}\operatorname{Im}(P_b^h v_h, v_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta}), \quad \text{as } h \to 0^+.$$

In the following, it will be convenient to identify $(\mathbb{R}^2)^*$ and $\mathbb{R}^2$ through the usual inner product. In particular, the cotangent bundle $T^*\mathbb{T}^2 = \mathbb{T}^2 \times (\mathbb{R}^2)^*$ will be identified with $\mathbb{T}^2 \times \mathbb{R}^2$.

5 Semiclassical measures

We denote by $\overline{T^*\mathbb{T}^2}$ the compactification of $T^*\mathbb{T}^2$ obtained by adding a point at infinity to each fiber (i.e., the set $\mathbb{T}^2 \times (\mathbb{R}^2 \cup \{\infty\})$). A neighbourhood of $(x, \infty) \in \overline{T^*\mathbb{T}^2}$ is a set $U \times (\{\infty\} \cup \mathbb{R}^2 \setminus K)$, where U is a neighbourhood of x in $\mathbb{T}^2$ and K a compact set in $\mathbb{R}^2$. Endowed with this topology, the set $\overline{T^*\mathbb{T}^2}$ is compact.

We denote by $S^0(T^*\mathbb{T}^2)$, S^0 for short, the space of functions $a(x, \xi)$ that satisfy the following properties:

1. $a \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$.
2. There exists a compact set $K \subset \mathbb{R}^2$ and a constant $k_0 \in \mathbb{C}$ such that $a(x, \xi) = k_0$ for all $\xi \in \mathbb{R}^2 \setminus K$.

Note that we have in particular $\mathscr{C}_c^\infty(T^*\mathbb{T}^2) \subset S^0(T^*\mathbb{T}^2)$.

To a symbol $a \in S^0(T^*\mathbb{T}^2)$, we associate its semiclassical Weyl quantization $\mathrm{Op}_h(a)$ by Formula (A.1), which, according to the Calderón-Vaillancourt Theorem (see Appendix A) defines a uniformly bounded operator on $L^2(\mathbb{T}^2)$.

From the sequence (v_h, h) (see for instance [GL93]), we can define (using again the Calderón-Vaillancourt Theorem) the associated Wigner distribution $V^h \in (S^0)'$ by

$$\langle V^h, a\rangle_{(S^0)', S^0} = (\mathrm{Op}_h(a)v_h, v_h)_{L^2(\mathbb{T}^2)}, \quad \text{for all } a \in S^0(T^*\mathbb{T}^2). \tag{5.1}$$

Decomposing v_h and a in Fourier series,

$$\hat{v}_h(k) = \frac{1}{2\pi}\int_{\mathbb{T}^2} e^{-ik\cdot x} v_h(x)dx, \quad \hat{a}(h,k,\xi) = \frac{1}{2\pi}\int_{\mathbb{T}^2} e^{-ik\cdot x} a(h,x,\xi)dx,$$

the expression (5.1) can be more explicitly rewritten as

$$\langle V^h, a\rangle_{(S^0)',S^0} = \frac{1}{2\pi}\sum_{k,j\in\mathbb{Z}^2} \hat{a}\left(h, j-k, \frac{h}{2}(k+j)\right)\hat{v}_h(k)\overline{\hat{v}_h(j)}.$$

Proposition 5.1. *The family (V^h) is bounded in $(S^0)'$. Hence, there exists a subsequence of the sequence (h, v_h) and an element $\mu \in (S^0)'$, such that $V^h \rightharpoonup \mu$ weakly in $(S^0)'$, i.e.*

$$(\mathrm{Op}_h(a)v_h, v_h)_{L^2(\mathbb{T}^2)} \to \langle \mu, a\rangle_{(S^0)',S^0} \quad \textit{for all } a \in S^0(T^*\mathbb{T}^2). \tag{5.2}$$

In addition, $\langle \mu, a\rangle_{(S^0)',S^0}$ is nonnegative if a is; in other words, μ may be identified with a nonnegative Radon measure on $\overline{T^*\mathbb{T}^2}$.

Notation: in what follows we shall denote by $\mathcal{M}^+(\overline{T^*\mathbb{T}^2})$ the set of nonnegative Radon measures on $\overline{T^*\mathbb{T}^2}$.

Proof. The proof is an adaptation from the original proof of Gérard [Gér91] (see also [GL93] in the semiclassical setting).

The fact that the Wigner distributions V^h are uniformly bounded in $(S^0)'$ follows from the Calderón-Vaillancourt theorem (see Appendix A), and from the boundedness of (v_h) in $L^2(\mathbb{T}^2)$.

The sharp Gårding inequality gives the existence of $C > 0$ such that, for all $a \geq 0$ and $h > 0$,

$$(\mathrm{Op}_h(a)v_h, v_h)_{L^2(\mathbb{T}^2)} \geq -Ch\|v_h\|^2_{L^2(\mathbb{T}^2)},$$

so that the distribution μ is nonnegative (and is hence a measure).

□

6 Zero-th and first order informations on μ

To simplify the notation, we set

$$P_b^h = P_0^h + ihb(x), \quad \text{with} \quad P_0^h = -h^2\Delta - 1 = \mathrm{Op}_h(|\xi|^2 - 1).$$

The geodesic flow on the torus $\phi_\tau : T^*\mathbb{T}^2 \to T^*\mathbb{T}^2$ for $\tau \in \mathbb{R}$ is the flow generated by the Hamiltonian vector field associated to the symbol $\frac{1}{2}(|\xi|^2 - 1)$, i.e. by the vector field $\xi \cdot \partial_x$ on $T^*\mathbb{T}^2$. Explicitely, we have

$$\phi_\tau(x,\xi) = (x + \tau\xi, \xi), \quad \tau \in \mathbb{R}, \quad (x,\xi) \in T^*\mathbb{T}^2.$$

Note that ϕ_τ preserves the ξ-component, and, in particular every energy layer $\{|\xi|^2 = C > 0\} \subset T^*\mathbb{T}^2$.

Now, we describe the first properties of the measure μ implied by (4.22).

We recall that for $\nu \in \mathscr{D}'(T^*\mathbb{T}^2)$, $(\phi_\tau)_*\nu \in \mathscr{D}'(T^*\mathbb{T}^2)$ is defined by $\langle (\phi_\tau)_*\nu, a\rangle = \langle \nu, a \circ \phi_\tau\rangle$ for all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$. In particular, $(\phi_\tau)_*\nu$ is a measure if ν is. We shall say that ν is an *invariant measure* if it is invariant by the geodesic flow, i.e. $(\phi_\tau)_*\nu = \nu$ for all $\tau \in \mathbb{R}$.

Proposition 6.1. *Let μ be as in Proposition 5.1. We have*

1. $\mathrm{supp}(\mu) \subset \{|\xi|^2 = 1\}$ *(hence is compact in $T^*\mathbb{T}^2$),*
2. $\mu(T^*\mathbb{T}^2) = 1$,
3. *μ is invariant by the geodesic flow, i.e. $(\phi_\tau)_*\mu = \mu$,*
4. *$\langle \mu, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)} = 0$, where $\mathcal{M}_c(T^*\mathbb{T}^2)$ denotes the space of compactly supported measures on $T^*\mathbb{T}^2$.*

In other words, μ is an invariant probability measure on $T^*\mathbb{T}^2$ vanishing on $\{b > 0\}$.

These are standard arguments, that we reproduce here for the reader's comfort. In particular, we recover all informations required to prove the Bardos-Lebeau-Rauch-Taylor uniform stabilization theorem under GCC. But we do not use here the second order informations of (4.22); this will be the key point to prove Theorem 2.6.

Proof. First, we take $\chi \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ depending only on the ξ variable, such that $\chi \geq 0$, $\chi(\xi) = 0$ for $|\xi| \leq 2$, and $\chi(\xi) = 1$ for $|\xi| \geq 3$. Hence, $\frac{\chi(\xi)}{|\xi|^2-1} \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ and we have the exact composition formula

$$\mathrm{Op}_h(\chi) = \mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) P_0^h,$$

since both operators are Fourier multipliers. Moreover, $\mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2-1}\right)$ is a bounded operator on $L^2(\mathbb{T}^2)$. As a consequence, we have

$$\left\langle V^h, \chi\right\rangle_{(S^0)', S^0} \to \langle \mu, \chi\rangle_{\mathcal{M}(\overline{T^*\mathbb{T}^2}), \mathscr{C}^0(\overline{T^*\mathbb{T}^2})},$$

together with

$$\begin{aligned}\left\langle V^h, \chi\right\rangle_{(S^0)', S^0} &= \left(\mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) P_0^h v_h, v_h\right)_{L^2(\mathbb{T}^2)} \\ &= \left(\mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) P_b^h v_h, v_h\right)_{L^2(\mathbb{T}^2)} - ih\left(\mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) b v_h, v_h\right)_{L^2(\mathbb{T}^2)}.\end{aligned}$$

Since $\|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(1)$ and $\|v_h\|_{L^2(\mathbb{T}^2)} = 1$, both terms in this expression vanish in the limit $h \to 0^+$. This implies that $\langle \mu, \chi\rangle_{\mathcal{M}(\overline{T^*\mathbb{T}^2}), \mathscr{C}^0(\overline{T^*\mathbb{T}^2})} = 0$. Since this holds for all χ as above, we have $\mathrm{supp}(\mu) \subset \{|\xi|^2 = 1\}$, which proves Item 1.

In particular, this implies that $\mu\left(\overline{T^*\mathbb{T}^2} \setminus T^*\mathbb{T}^2\right) = 0$. Now, Item 2 is a direct consequence of $1 = \|v_h\|^2_{L^2(\mathbb{T}^2)} \to \langle \mu, 1\rangle_{\mathcal{M}(\overline{T^*\mathbb{T}^2}), \mathscr{C}^0(\overline{T^*\mathbb{T}^2})}$ and Item 1. Item 4 is a direct consequence of $(bv_h, v_h)_{L^2(\mathbb{T}^2)} = o(1)$.

Finally, for $a \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2)$, we recall that

$$\left[P_0^h, \mathrm{Op}_h(a)\right] = \frac{h}{i}\mathrm{Op}_h(\{|\xi|^2 - 1, a\}) = \frac{2h}{i}\mathrm{Op}_h(\xi \cdot \partial_x a),$$

is a consequence of the Weyl quantization (any other quantization would have left an error term of order $\mathcal{O}(h^2)$). Hence, (5.1) yields

$$\left\langle V^h, \xi \cdot \partial_x a\right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}^\infty_c(T^*\mathbb{T}^2)} \to \langle \mu, \xi \cdot \partial_x a\rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}^0_c(T^*\mathbb{T}^2)}, \tag{6.1}$$

together with

$$\begin{aligned}\left\langle V^h, \xi \cdot \partial_x a\right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}^\infty_c(T^*\mathbb{T}^2)} &= \frac{i}{2h}\left(\left[P_0^h, \mathrm{Op}_h(a)\right] v_h, v_h\right)_{L^2(\mathbb{T}^2)} \\ &= \frac{i}{2h}\left(\mathrm{Op}_h(a) v_h, P_0^h v_h\right)_{L^2(\mathbb{T}^2)} - \frac{i}{2h}\left(\mathrm{Op}_h(a) P_0^h v_h, v_h\right)_{L^2(\mathbb{T}^2)} \\ &= \frac{i}{2h}\left(\mathrm{Op}_h(a) v_h, P_b^h v_h\right)_{L^2(\mathbb{T}^2)} - \frac{i}{2h}\left(\mathrm{Op}_h(a) P_b^h v_h, v_h\right)_{L^2(\mathbb{T}^2)} \\ &\quad - \frac{1}{2}\left(\mathrm{Op}_h(a) v_h, b v_h\right)_{L^2(\mathbb{T}^2)} - \frac{1}{2}\left(\mathrm{Op}_h(a) b v_h, v_h\right)_{L^2(\mathbb{T}^2)}.\end{aligned} \tag{6.2}$$

In this expression, we have $\frac{1}{h}\left(\mathrm{Op}_h(a) v_h, P_b^h v_h\right)_{L^2(\mathbb{T}^2)} \to 0$ and $\frac{1}{h}\left(\mathrm{Op}_h(a) P_b^h v_h, v_h\right)_{L^2(\mathbb{T}^2)} \to 0$ since $\|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(h)$. Moreover, the last two terms can be estimated by

$$\left|\left(\mathrm{Op}_h(a) b v_h, v_h\right)_{L^2(\mathbb{T}^2)}\right| \leq \|\sqrt{b} v_h\|_{L^2(\mathbb{T}^2)} \|\sqrt{b}\,\mathrm{Op}_h(a) v_h\|_{L^2(\mathbb{T}^2)} = o(1), \tag{6.3}$$

since $(bv_h, v_h)_{L^2(\mathbb{T}^2)} = o(1)$. This yields $\left\langle V^h, \xi \cdot \partial_x a \right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}_c^\infty(T^*\mathbb{T}^2)} \to 0$, so that, using (6.1), $\langle \mu, \xi \cdot \partial_x a \rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}_c^0(T^*\mathbb{T}^2)} = 0$ for all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$. Replacing a by $a \circ \phi_\tau$ and integrating with respect to the parameter τ gives $(\phi_\tau)_* \mu = \mu$, which concludes the proof of Item 3.

□

7 Geometry on the torus and decomposition of invariant measures

7.1 Resonant and non-resonant vectors on the torus

In this section, we collect several facts concerning the geometry of $T^*\mathbb{T}^2$ and its resonant subspaces. Most of the setting and the notation comes from [AM11, Section 2].

We shall say that a submodule $\Lambda \subset \mathbb{Z}^2$ is primitive if $\langle \Lambda \rangle \cap \mathbb{Z}^2 = \Lambda$, where $\langle \Lambda \rangle$ denotes the linear subspace of $\mathbb{R}^2$ spanned by Λ. The family of all primitive submodules will be denoted by $\mathcal{P}$.

Let us denote by $\Omega_j \subset \mathbb{R}^2$, for $j = 0, 1, 2$, the set of resonant vectors of order exactly j, i.e.,

$$\Omega_j := \{\xi \in \mathbb{R}^2 \text{ such that } \operatorname{rk}(\Lambda_\xi) = 2 - j\}, \quad \text{with} \quad \Lambda_\xi := \left\{k \in \mathbb{Z}^2 \text{ such that } \xi \cdot k = 0\right\} = \xi^\perp \cap \mathbb{Z}^2.$$

Note that the sets Ω_j form a partition of $\mathbb{R}^2$, and that we have

- $\Omega_0 = \{0\}$;
- $\xi \in \Omega_1$ if and only if the geodesic issued from any $x \in \mathbb{T}^2$ in the direction ξ is periodic;
- $\xi \in \Omega_2$ if and only if the geodesic issued from any $x \in \mathbb{T}^2$ in the direction ξ is dense in $\mathbb{T}^2$.

For each $\Lambda \in \mathcal{P}$ such that $\operatorname{rk}(\Lambda) = 1$, we define

$$\begin{gathered} \Lambda^\perp := \left\{\xi \in \mathbb{R}^2 \text{ such that } \xi \cdot k = 0 \text{ for all } k \in \Lambda\right\}, \\ \mathbb{T}_\Lambda := \langle \Lambda \rangle / 2\pi\Lambda, \\ \mathbb{T}_{\Lambda^\perp} := \Lambda^\perp / (2\pi \mathbb{Z}^2 \cap \Lambda^\perp). \end{gathered}$$

Note that $\mathbb{T}_\Lambda$ and $\mathbb{T}_{\Lambda^\perp}$ are two submanifolds of $\mathbb{T}^2$ diffeomorphic to one-dimensional tori. Their cotangent bundles admit the global trivialisations $T^*\mathbb{T}_\Lambda = \mathbb{T}_\Lambda \times \langle \Lambda \rangle$ and $T^*\mathbb{T}_{\Lambda^\perp} = \mathbb{T}_{\Lambda^\perp} \times \Lambda^\perp$.

For a function f on $\mathbb{T}^2$ with Fourier coefficients $(\hat{f}(k))_{k \in \mathbb{Z}^2}$, and $\Lambda \in \mathcal{P}$, we shall say that f has only Fourier modes in Λ if $\hat{f}(k) = 0$ for $k \notin \Lambda$. This means that f is constant in the direction $\Lambda^\perp$, or, equivalently, that $\sigma \cdot \partial_x f = 0$ for all $\sigma \in \Lambda^\perp$. We denote by $L^p_\Lambda(\mathbb{T}^2)$ the subspace of $L^p(\mathbb{T}^2)$ consisting of functions having only Fourier modes in Λ. For a function $f \in L^2(\mathbb{T}^2)$ (resp. a symbol $a \in S^0(T^*\mathbb{T}^2)$), we denote by $\langle f \rangle_\Lambda$ its orthogonal projection on $L^2_\Lambda(\mathbb{T}^2)$, i.e. the average of f along $\Lambda^\perp$:

$$\langle f \rangle_\Lambda (x) := \sum_{k \in \Lambda} \frac{e^{ik \cdot x}}{2\pi} \hat{f}(k) \qquad \left(\text{resp. } \langle a \rangle_\Lambda (x, \xi) := \sum_{k \in \Lambda} \frac{e^{ik \cdot x}}{2\pi} \hat{a}(k, \xi)\right).$$

If $\operatorname{rk}(\Lambda) = 1$ and v is a vector in $\Lambda^\perp \setminus \{0\}$, we also have

$$\langle f \rangle_\Lambda (x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x + tv) dt. \tag{7.1}$$

In particular, note that $\langle f \rangle_\Lambda$ (resp. $\langle a \rangle_\Lambda$) is nonnegative if f (resp. a) is, and that $\langle f \rangle_\Lambda \in \mathscr{C}^\infty(\mathbb{T}^2)$ (resp. $\langle a \rangle_\Lambda \in S^0(T^*\mathbb{T}^2)$) if $f \in \mathscr{C}^\infty(\mathbb{T}^2)$ (resp. $a \in S^0(T^*\mathbb{T}^2)$).

Finally, given $f \in L^\infty_\Lambda(\mathbb{T}^2)$, we denote by m_f the bounded operator on $L^2_\Lambda(\mathbb{T}^2)$, consisting in the multiplication by f.

7.2 Decomposition of invariant measures

We denote by $\mathcal{M}^+(T^*\mathbb{T}^2)$ the set of finite, nonnegative measures on $T^*\mathbb{T}^2$. With the definitions above, we have the following decomposition Lemmata, proved in [Mac10] or [AM11, Section 2]. These properties are given for general measures $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$. Of course, they apply in particular to the measure μ defined by Proposition 5.1.

Lemma 7.1. *Let $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$. Then μ decomposes as a sum of nonnegative measures*

$$\mu = \mu|_{\mathbb{T}^2\times\{0\}} + \mu|_{\mathbb{T}^2\times\Omega_2} + \sum_{\Lambda\in\mathcal{P},\mathrm{rk}(\Lambda)=1} \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})} \tag{7.2}$$

Given $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$, we define its Fourier coefficients by the complex measures on $\mathbb{R}^2$:

$$\hat{\mu}(k,\cdot) := \int_{\mathbb{T}^2} \frac{e^{-ik\cdot x}}{2\pi}\mu(dx,\cdot), \quad k\in\mathbb{Z}.$$

One has, in the sense of distributions, the following Fourier inversion formula:

$$\mu(x,\xi) = \sum_{k\in\mathbb{Z}^2} \frac{e^{ik\cdot x}}{2\pi}\hat{\mu}(k,\xi).$$

Lemma 7.2. *Let $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$ and $\Lambda \in \mathcal{P}$. Then, the distribution*

$$\langle\mu\rangle_\Lambda(x,\xi) := \sum_{k\in\Lambda} \frac{e^{ik\cdot x}}{2\pi}\hat{\mu}(k,\xi),$$

is in $\mathcal{M}^+(T^*\mathbb{T}^2)$ and satisfies, for all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$,

$$\langle\langle\mu\rangle_\Lambda, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} = \langle\mu,\langle a\rangle_\Lambda\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)}.$$

Lemma 7.3. *Let $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$ be an* invariant *measure. Then, for all $\Lambda\in\mathcal{P}$, $\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$ is also a nonnegative* invariant *measure and*

$$\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})} = \langle\mu\rangle_\Lambda|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}.$$

Let us now come back to the measure μ given by Proposition 5.1, which satisfies all properties listed in Proposition 6.1. In particular, this measure vanishes on the non-empty open subset of $\mathbb{T}^2$ given by $\{b > 0\}$ (see Item 4 in Proposition 6.1). As a consequence of Proposition 6.1, and of the three lemmata above, this yields the following lemma.

Lemma 7.4. *We have $\mu = \sum_{\Lambda\in\mathcal{P},\mathrm{rk}(\Lambda)=1} \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$.*

As a consequence of Proposition 6.1, we have indeed that the measure μ is supported in $\{|\xi| = 1\}$, which implies $\mu|_{\mathbb{T}^2\times\{0\}} = 0$. In addition, Lemma 7.3 applied with $\Lambda = \{0\}$ implies that $\mu|_{\mathbb{T}^2\times\Omega_2}$ is constant in x – and thus vanishes everywhere since it vanishes on $\{b > 0\}$.

Remark 7.5. Since the measure μ is supported in $\{|\xi| = 1\}$ (Proposition 6.1, Item 1), we have

$$\mu|_{\mathbb{T}^2\times\Lambda^\perp} = \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$$

(which simplifies the notation).

As a consequence of these lemmata and the last remark, the study of the measure μ is now reduced to that of all nonnegative invariant measures $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ with $\mathrm{rk}(\Lambda) = 1$.

The aim of the next sections is to prove that the measure $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ vanishes identically, for each periodic direction $\Lambda^\perp$.

7.3 Geometry of the subtori $\mathbb{T}_\Lambda$ and $\mathbb{T}_{\Lambda^\perp}$

To study the measure $\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$, we need to describe briefly the geometry of the subtori $\mathbb{T}_\Lambda$ and $\mathbb{T}_{\Lambda^\perp}$ of $\mathbb{T}^2$, and introduce adapted coordinates.

We define χ_Λ the linear isomorphism

$$\chi_\Lambda : \Lambda^\perp \times \langle\Lambda\rangle \to \mathbb{R}^2 : \quad (s,y) \mapsto s+y,$$

and denote by $\tilde{\chi}_\Lambda : T^*\Lambda^\perp \times T^*\langle\Lambda\rangle \to T^*\mathbb{R}^2$ its extension to the cotangent bundle. This application can be defined as follows: for $(s,\sigma) \in T^*\Lambda^\perp = \Lambda^\perp \times (\Lambda^\perp)^*$ and $(y,\eta) \in T^*\langle\Lambda\rangle = \langle\Lambda\rangle \times \langle\Lambda\rangle^*$, we can extend σ to a covector of $\mathbb{R}^2$ vanishing on $\langle\Lambda\rangle$ and η to a covector of $\mathbb{R}^2$ vanishing on $\Lambda^\perp$. Remember that we identify $(\mathbb{R}^2)^*$ with $\mathbb{R}^2$ through the usual inner product; thus we can also see σ as an element of $\Lambda^\perp$ and η as an element of $\langle\Lambda\rangle$. Then, we have

$$\tilde{\chi}_\Lambda(s,\sigma,y,\eta) = (s+y, \sigma+\eta) \in T^*\mathbb{R}^2 = \mathbb{R}^2 \times (\mathbb{R}^2)^*.$$

Conversely, any $\xi \in (\mathbb{R}^2)^*$ can be decomposed into $\xi = \sigma + \eta$ where $\sigma \in \Lambda^\perp$ and $\eta \in \langle\Lambda\rangle$. We denote by P_Λ the orthogonal projection of $\mathbb{R}^2$ onto $\langle\Lambda\rangle$, i.e. $P_\Lambda\xi = \eta$.

Next, the map χ_Λ goes to the quotient, giving a smooth Riemannian covering of $\mathbb{T}^2$ by

$$\pi_\Lambda : \mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda \to \mathbb{T}^2 : \quad (s,y) \mapsto s+y.$$

We shall denote by $\tilde{\pi}_\Lambda$ its extension to cotangent bundles:

$$\tilde{\pi}_\Lambda : T^*\mathbb{T}_{\Lambda^\perp} \times T^*\mathbb{T}_\Lambda \to T^*\mathbb{T}^2.$$

As the map π_Λ is not an injection (because the torus $\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda$ contains several copies of $\mathbb{T}^2$), we introduce its degree p_Λ, which is also equal to $\frac{\mathrm{Vol}(\mathbb{T}_{\Lambda^\perp}\times\mathbb{T}_\Lambda)}{\mathrm{Vol}(\mathbb{T}^2)}$.

Then, the application

$$T_\Lambda u := \frac{1}{\sqrt{p_\Lambda}} u \circ \chi_\Lambda,$$

defines a linear isomorphism $L^2_{\mathrm{loc}}(\mathbb{R}^2) \to L^2_{\mathrm{loc}}(\Lambda^\perp \times \langle\Lambda\rangle)$. Note that because of the factor $\frac{1}{\sqrt{p_\Lambda}}$, T_Λ maps $L^2(\mathbb{T}^2)$ isometrically into a subspace of $L^2(\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda)$. Moreover, T_Λ maps $L^2_\Lambda(\mathbb{T}^2)$ into $L^2(\mathbb{T}_\Lambda) \subset L^2(\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda)$, since the nonvanishing Fourier modes of $u \in L^2_\Lambda(\mathbb{T}^2)$ correspond only to frequencies $k \in \Lambda$. This reads

$$T_\Lambda u(s,y) = \frac{1}{\sqrt{p_\Lambda}} u(y) \text{ for } (s,y) \in \mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda. \tag{7.3}$$

Since $\tilde{\chi}_\Lambda$ is linear, we have, for any $a \in \mathscr{C}^\infty(T^*\mathbb{R}^2)$

$$T_\Lambda \operatorname{Op}_h(a) = \operatorname{Op}_h(a \circ \tilde{\chi}_\Lambda) T_\Lambda, \tag{7.4}$$

where on the left Op_h is the Weyl quantization on $\mathbb{R}^2$ (A.1), and on the right Op_h is the Weyl quantization on $\Lambda^\perp \times \langle\Lambda\rangle$. Next, we denote by $\operatorname{Op}_h^{\Lambda^\perp}$ and $\operatorname{Op}_h^\Lambda$ the Weyl quantization operators defined on smooth test functions on $T^*\Lambda^\perp \times T^*\langle\Lambda\rangle$ and acting only on the variables in $T^*\Lambda^\perp$ and $T^*\langle\Lambda\rangle$ respectively, leaving the other frozen. For any $a \in \mathscr{C}_c^\infty(T^*\Lambda^\perp \times T^*\langle\Lambda\rangle)$, we have :

$$\operatorname{Op}_h(a) = \operatorname{Op}_h^{\Lambda^\perp} \circ \operatorname{Op}_h^\Lambda(a) = \operatorname{Op}_h^\Lambda \circ \operatorname{Op}_h^{\Lambda^\perp}(a). \tag{7.5}$$

Now, if the symbol $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$ has only Fourier modes in Λ, we remark, in view of (7.3), that $a \circ \tilde{\pi}_\Lambda$ does not depend on $s \in \mathbb{T}_{\Lambda^\perp}$. Therefore, we sometimes write $a \circ \tilde{\pi}_\Lambda(\sigma,y,\eta)$ for $a \circ \tilde{\pi}_\Lambda(s,\sigma,y,\eta)$ and (7.4)-(7.5) give

$$T_\Lambda \operatorname{Op}_h(a) = \operatorname{Op}_h^\Lambda \circ \operatorname{Op}_h^{\Lambda^\perp}(a \circ \tilde{\pi}_\Lambda) T_\Lambda = \operatorname{Op}_h^\Lambda(a \circ \tilde{\pi}_\Lambda(hD_s,\cdot,\cdot)) T_\Lambda. \tag{7.6}$$

Note that for every $\sigma \in \Lambda^\perp$, the operator $\operatorname{Op}_h^\Lambda(a \circ \tilde{\pi}_\Lambda(\sigma,\cdot,\cdot))$ maps $L^2(\mathbb{T}_\Lambda)$ into itself. More precisely, it maps the subspace $T_\Lambda(L^2_\Lambda(\mathbb{T}^2))$ into itself.

8 Change of quasimode and construction of an invariant cut-off function

In this section, we first construct from the quasimode v_h another quasimode w_h, that will be easier to handle when studying the measure $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$. Indeed w_h is basically a microlocalization of v_h in the direction $\Lambda^\perp$ at a precise concentration rate.

Moreover, we introduce a cutoff function $\chi_h^\Lambda(x) = \chi_h^\Lambda(y,s)$, well-adapted to the damping coefficient b and to the invariance of the measure $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ in the direction $\Lambda^\perp$ (this cutoff function plays the role of the function $\chi(b/h)$ used in [BH07] in the case where b is itself invariant in the direction $\Lambda^\perp$). Its construction is a key point in the proof of Theorem 2.6.

Let $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ be a nonnegative function such that $\chi = 1$ in a neighbourhood of the origin. We first define

$$w_h := \mathrm{Op}_h\left(\chi\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right) v_h,$$

which implicitely depends on $\alpha \in (0,1)$. The following lemma implies that, for δ and α sufficiently small, w_h is as well a $o(h^{2+\delta})$-quasimode for P_b^h.

Lemma 8.1. *For any $\alpha > 0$ such that*

$$\delta + \frac{\varepsilon}{2} + \alpha \leq \frac{1}{2}, \qquad 3\alpha + 2\delta < 1, \tag{8.1}$$

we have

$$\|P_b^h w_h\|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta}).$$

As a consequence of this lemma, the semiclassical measures associated to w_h satisfy in particular the conclusions of Proposition 6.1. Moreover, the following proposition implies that the sequence w_h contains all the information in the direction $\Lambda^\perp$.

Proposition 8.2. *For any $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$ and any $\alpha \in (0, 3/4)$ satisfying the assumptions of Lemma 8.1, we have*

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} = \lim_{h\to 0} \left(\mathrm{Op}_h(a) w_h, w_h\right)_{L^2(\mathbb{T}^2)}.$$

Next, we state the desired properties of the cutoff function χ_h^Λ. The proof of its existence is a crucial point in the proof of Theorem 2.6.

Proposition 8.3. *For $\delta = 8\varepsilon$, and $\varepsilon < \frac{1}{76}$, there exists α satisfying* (8.1)*, such that for any constant $c_0 > 0$, there exists a cutoff function $\chi_h^\Lambda \in \mathscr{C}^\infty(\mathbb{T}^2)$ valued in $[0,1]$, such that*

1. *$\chi_h^\Lambda = \chi_h^\Lambda(y)$ does not depend on the variable s (i.e. χ_h^Λ is $\Lambda^\perp$-invariant),*
2. *$\|(1-\chi_h^\Lambda) w_h\|_{L^2(\mathbb{T}^2)} = o(1)$,*
3. *$b \leq c_0 h$ on $\mathrm{supp}(\chi_h^\Lambda)$,*
4. *$\|\partial_y \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$,*
5. *$\|\partial_y^2 \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$.*

Note that the function χ_h^Λ implicitely depends on the constant c_0, that will be taken arbitrarily small in Section 10.

In the particular case where the damping function b is invariant in one direction, this proposition is not needed. In this case, one can take as in [BH07] $\chi_h^\Lambda = \chi(\frac{b}{c_0 h})$. In the d-dimensional torus, this cutoff functions works as well if b is invariant in $d-1$ directions, and an analogue of Theorem 2.6 can be stated in this setting. Unfortunately, our construction of the function χ_h^Λ (see the proof of Proposition 8.3 in Section 13) strongly relies on the fact that all trapped directions are periodic, and fails in higher dimensions.

We give here a proof of Lemma 8.1. Because of their technicality, we postpone the proofs of Propositions 8.2 and 8.3 to Sections 12 and 13 respectively.

Proof of Lemma 8.1. First, we develop

$$P_b^h w_h = P_b^h \operatorname{Op}_h\left(\chi\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right) v_h = \operatorname{Op}_h\left(\chi\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right) P_b^h v_h + ih\left[b, \operatorname{Op}_h\left(\chi\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right)\right] v_h, \tag{8.2}$$

since P_0^h and $\operatorname{Op}_h\left(\chi\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right)$ are both Fourier mutipliers. We know that

$$\left\|\operatorname{Op}_h\left(\chi\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right) P_b^h v_h\right\|_{L^2(\mathbb{T}^2)} \leq \|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta}).$$

It only remains to study the operator

$$\left[b, \operatorname{Op}_h\left(\chi\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right)\right] = ih^{1-\alpha} \operatorname{Op}_h\left(\partial_y b\, \chi'\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right) + \mathcal{O}_{\mathcal{L}(L^2)}(h^{2(1-\alpha)}) \tag{8.3}$$

according to the symbolic calculus. Moreover, using Assumption (2.12), we have

$$\left|\partial_y b\, \chi'\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right| \leq C b^{1-\varepsilon}.$$

The sharp Gårding inequality applied to the nonnegative symbol

$$C^2 b^{2(1-\varepsilon)} - \left|\partial_y b\, \chi'\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right|^2,$$

then yields

$$\left(\operatorname{Op}_h\left(C^2 b^{2(1-\varepsilon)} - \left|\partial_y b\, \chi'\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right|^2\right) v_h, v_h\right)_{L^2(\mathbb{T}^2)} \geq -Ch^{1-\alpha},$$

and hence

$$\left\|\operatorname{Op}_h\left(\partial_y b\, \chi'\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right) v_h\right\|_{L^2(\mathbb{T}^2)}^2 \leq C^2 (b^{2(1-\varepsilon)} v_h, v_h)_{L^2(\mathbb{T}^2)} + O(h^{1-\alpha}). \tag{8.4}$$

When using the inequality $\int f^{1-\varepsilon} d\nu \leq \left(\int f d\nu\right)^{1-\varepsilon}$ for nonnegative functions (with $d\nu = |v_h(x)|^2 dx$), we obtain

$$(b^{2(1-\varepsilon)} v_h, v_h)_{L^2(\mathbb{T}^2)} \leq (b^2 v_h, v_h)_{L^2(\mathbb{T}^2)}^{(1-\varepsilon)} \leq C\|b v_h\|_{L^2(\mathbb{T}^2)}^{2(1-\varepsilon)} = o(h^{1-\varepsilon}).$$

Combining this estimate together with (8.3) and (8.4) gives

$$\left\|ih\left[b, \operatorname{Op}_h\left(\chi\left(\frac{|P_\Lambda \xi|}{h^\alpha}\right)\right)\right] v_h\right\|_{L^2(\mathbb{T}^2)} = o(h^{\frac{5}{2}-\alpha-\frac{\varepsilon}{2}}) + O(h^{\frac{5-3\alpha}{2}}).$$

Coming back to the expression of $P_b^h w_h$ given in (8.2), this concludes the proof of Lemma 8.1. □

9 Second microlocalization on a resonant affine subspace

We want to analyse precisely the structure of the restriction $\mu|_{\mathbb{T}^2 \times (\Lambda^\perp \setminus \{0\})}$, using the full information contained in $o(h^{2+\delta})$-quasimodes like v_h and w_h.

From now on, we want to take advantage of the family w_h of $o(h^{2+\delta})$-quasimodes constructed in Section 8, which are microlocalised in the direction $\Lambda^\perp$. Hence, we define the Wigner distribution $W^h \in \mathscr{D}'(T^*\mathbb{T}^2)$ associated to the functions w_h and the scale h, by

$$\left\langle W^h, a\right\rangle_{(S^0)', S^0} = (\operatorname{Op}_h(a) w_h, w_h)_{L^2(\mathbb{T}^2)} \quad \text{for all } a \in S^0(T^*\mathbb{T}^2).$$

According to Proposition 8.2, we recover in the limit $h \to 0$,

$$\left\langle W^h, a \right\rangle_{(S^0)', S^0} \to \left\langle \mu|_{\mathbb{T}^2 \times \Lambda^\perp}, a \right\rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}_c^0(T^*\mathbb{T}^2)},$$

for any $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$ (and α satisfying (8.1)).

To provide a precise study of $\mu|_{\mathbb{T}^2 \times \Lambda^\perp}$, we shall introduce as in [Mac10, AM11] two-microlocal semiclassical measures, describing at a finer level the concentration of the sequence v_h on the resonant subspace

$$\Lambda^\perp = \{\xi \in \mathbb{R}^2 \text{ such that } P_\Lambda \xi = 0\}.$$

These objects have been introduced in the local Euclidean case by Nier [Nie96] and Fermanian-Kammerer [FK00b, FK00a]. A specific concentration scale may also be chosen in the in the two-microlocal variable, giving rise to the two-scales semiclassical measures studied by Miller [Mil96, Mil97] and Fermanian-Kammerer and Gérard [FKG02].

We first have to describe the adapted symbol class (inspired by [FK00a] and used in [AM11]). According to Lemma 7.3 (see also Remark 7.5), it suffices to test the measure $\mu|_{\mathbb{T}^2 \times \Lambda^\perp}$ with functions constant in the direction $\Lambda^\perp$ (or equivalently, having only x-Fourier modes in Λ, in the sense of the following definition).

Definition 9.1. Given $\Lambda \in \mathcal{P}$, we shall say that $a \in S^1_\Lambda$ if $a = a(x, \xi, \eta) \in \mathscr{C}^\infty(T^*\mathbb{T}^2 \times \langle\Lambda\rangle)$ and

1. there exists a compact set $K_a \subset T^*\mathbb{T}^2$ such that, for all $\eta \in \langle\Lambda\rangle$, the function $(x, \xi) \mapsto a(x, \xi, \eta)$ is compactly supported in K_a;

2. a is homogeneous of order zero at infinity in the variable $\eta \in \langle\Lambda\rangle$; i.e., if we denote by $\mathbb{S}_\Lambda := \mathbb{S}^1 \cap \langle\Lambda\rangle$ the unit sphere in $\langle\Lambda\rangle$, there exists $R_0 > 0$ (depending on a) and $a_{\mathrm{hom}} \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda)$ such that
$$a(x, \xi, \eta) = a_{\mathrm{hom}}\left(x, \xi, \frac{\eta}{|\eta|}\right), \quad \text{for} \quad |\eta| \geq R_0 \text{ and } (x, \xi) \in T^*\mathbb{T}^2;$$
for $\eta \neq 0$, we will also use the notation $a(x, \xi, \infty\eta) := a_{\mathrm{hom}}\left(x, \xi, \frac{\eta}{|\eta|}\right)$.

3. a has only x-Fourier modes in Λ, i.e.
$$a(x, \xi, \eta) = \sum_{k \in \Lambda} \frac{e^{ik\cdot x}}{2\pi} \hat{a}(k, \xi, \eta).$$

Note that this last assumption is equivalent to saying that $\sigma \cdot \partial_x a = 0$ for any $\sigma \in \Lambda^\perp$. We denote by ${S^1_\Lambda}'$ the topological dual space of S^1_Λ.

Let $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ be a nonnegative function such that $\chi = 1$ in a neighbourhood of the origin. Let $R > 0$. The previous remark allows us to define, for $a \in S^1_\Lambda$ the two following elements of ${S^1_\Lambda}'$:

$$\left\langle W_R^{h,\Lambda}, a \right\rangle_{{S^1_\Lambda}', S^1_\Lambda} := \left\langle W^h, \left(1 - \chi\left(\frac{|P_\Lambda \xi|}{Rh}\right)\right) a\left(x, \xi, \frac{P_\Lambda \xi}{h}\right) \right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}_c^\infty(T^*\mathbb{T}^2)}, \tag{9.1}$$

$$\left\langle W^h_{R,\Lambda}, a \right\rangle_{{S^1_\Lambda}', S^1_\Lambda} := \left\langle W^h, \chi\left(\frac{|P_\Lambda \xi|}{Rh}\right) a\left(x, \xi, \frac{P_\Lambda \xi}{h}\right) \right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}_c^\infty(T^*\mathbb{T}^2)}. \tag{9.2}$$

In particular, for any $R > 0$ and $a \in S^1_\Lambda$, we have

$$\left\langle W^h, a\left(x, \xi, \frac{P_\Lambda \xi}{h}\right) \right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}_c^\infty(T^*\mathbb{T}^2)} = \left\langle W_R^{h,\Lambda}, a \right\rangle_{{S^1_\Lambda}', S^1_\Lambda} + \left\langle W^h_{R,\Lambda}, a \right\rangle_{{S^1_\Lambda}', S^1_\Lambda}. \tag{9.3}$$

The following two propositions are the analogues of [FK00a] in our context. They state the existence of the two-microlocal semiclassical measures, as the limit objects of $W_R^{h,\Lambda}$ and $W^h_{R,\Lambda}$.

Proposition 9.2. *There exists a subsequence (h, w_h) and a nonnegative measure $\nu^\Lambda \in \mathcal{M}^+(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda)$ such that, for all $a \in S^1_\Lambda$, we have*

$$\lim_{R\to\infty} \lim_{h\to 0} \left\langle W_R^{h,\Lambda}, a \right\rangle_{{S^1_\Lambda}', S^1_\Lambda} = \left\langle \nu^\Lambda, a_{\mathrm{hom}}\left(x, \xi, \frac{\eta}{|\eta|}\right) \right\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda), \mathscr{C}^0_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}.$$

To define the limit of the distributions $W^h_{R,\Lambda}$, we need first to introduce operator spaces and operator-valued measures, following [Gér91]. Given a Hilbert space H (in the following, we shall use $H = L^2(\mathbb{T}_\Lambda)$), we denote respectively by $\mathcal{K}(H)$, $\mathcal{L}^1(H)$ the spaces of compact and trace class operators on H. We recall that they are both two-sided ideals of the ring $\mathcal{L}(H)$ of bounded operators on H. We refer for instance to [RS80, Chapter VI.6] for a description of the space $\mathcal{L}^1(H)$ and its basic properties. Given a Polish space T (in the following, we shall use $T = T^*\mathbb{T}_{\Lambda^\perp}$), we denote by $\mathcal{M}^+(T; \mathcal{L}^1(H))$ the space of nonnegative measures on T, taking values in $\mathcal{L}^1(H)$. More precisely, we have $\rho \in \mathcal{M}^+(T; \mathcal{L}^1(H))$ if ρ is a bounded linear form on $\mathscr{C}^0_c(T)$ such that, for every nonnegative function $a \in \mathscr{C}^0_c(T)$, $\langle \rho, a\rangle_{\mathcal{M}(T), \mathscr{C}^0_c(T)} \in \mathcal{L}^1(H)$ is a nonnegative hermitian operator. As a consequence of [RS80, Theorem VI.26], these measures can be identified in a natural way to nonnegative linear functionals on $\mathscr{C}^0_c(T; \mathcal{K}(H))$.

Proposition 9.3. *There exists a subsequence (h, w_h) and a nonnegative measure*

$$\rho_\Lambda \in \mathcal{M}^+(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{L}^1(L^2(\mathbb{T}_\Lambda))),$$

such that, for all $K \in \mathscr{C}^\infty_c(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{K}(L^2(\mathbb{T}_\Lambda)))$,

$$\lim_{h\to 0} \left(K(s, hD_s)T_\Lambda w_h, T_\Lambda w_h\right)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} = \mathrm{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} K(s,\sigma)\rho_\Lambda(ds, d\sigma)\right\}. \tag{9.4}$$

Moreover (for the same subsequence), for all $a \in S^1_\Lambda$, we have

$$\lim_{R\to\infty}\lim_{h\to 0} \langle W^h_{R,\Lambda}, a\rangle_{{S^1_\Lambda}', S^1_\Lambda} = \mathrm{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} \mathrm{Op}^\Lambda_1\left(a(\tilde{\pi}_\Lambda(\sigma, y, 0), \eta)\right)\rho_\Lambda(ds, d\sigma)\right\}. \tag{9.5}$$

In the left hand-side of (9.4), the inner product actually means

$$\begin{aligned}(K(s, hD_s)&T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp} L^2(\mathbb{T}_\Lambda))} \\ &= \int_{s\in\mathbb{T}_{\Lambda^\perp}, s'\in\Lambda^\perp, \sigma\in\Lambda^\perp} e^{\frac{i}{h}(s-s')\cdot\sigma}\left(K\left(\frac{s+s'}{2}, \sigma\right)T_\Lambda w_h(s', y), T_\Lambda w_h(s,y)\right)_{L^2_y(\mathbb{T}_\Lambda)} ds\, ds'\, d\sigma.\end{aligned}$$

In the expression (9.5), remark that for each $\sigma \in \Lambda^\perp$, the operator $\mathrm{Op}^\Lambda_1\left(a(\tilde{\pi}_\Lambda(\sigma, y, 0), \eta\right)$ is in $\mathcal{L}(L^2(\mathbb{T}_\Lambda))$. Hence, its product with the operator $\rho_\Lambda(ds, d\sigma)$ defines a trace-class operator.

Before proving Propositions 9.2 and 9.3, we explain how to reconstruct the measure $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ from the two-microlocal measures ν^Λ and ρ_Λ. This reduces the study of the measure μ to that of all two-microlocal measures ν^Λ and ρ_Λ, for $\Lambda \in \mathcal{P}$.

We denote by $\mathcal{M}^+_c(T)$ the set of compactly supported measures on T, and by $\langle\cdot,\cdot\rangle_{\mathcal{M}_c(T), \mathscr{C}^0(T)}$ the associated duality bracket.

Proposition 9.4. *For all $a \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2)$ having only x-Fourier modes in Λ (i.e. for all $a \in S^1_\Lambda$ independent of the third variable $\eta \in \langle\Lambda\rangle$), we have*

$$\langle \mu, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}^0_c(T^*\mathbb{T}^2)} = \left\langle \nu^\Lambda, a\right\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda), \mathscr{C}^0_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} + \mathrm{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{a\circ\tilde{\pi}_\Lambda}(\sigma)\rho_\Lambda(ds, d\sigma)\right\}, \tag{9.6}$$

and

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} = \langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} + \operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{a\circ\tilde{\pi}_\Lambda}(\sigma)\rho_\Lambda(ds,d\sigma)\right\}, \tag{9.7}$$

where for $\sigma \in \Lambda^\perp$, $m_{a\circ\tilde{\pi}_\Lambda}(\sigma)$ denotes the multiplication in $L^2(\mathbb{T}_\Lambda)$ by the function $y \mapsto a\circ\tilde{\pi}_\Lambda(\sigma, y)$.

Moreover, we have $\nu^\Lambda \in \mathcal{M}_c^+(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)$ and $\rho_\Lambda \in \mathcal{M}_c^+(T^*\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))$ (i.e. both measures are compactly supported).

Formula (9.7) follows immediately from (9.6) by restriction. By the definition of the measure ρ_Λ, we see that it is already supported on $\mathbb{T}^2\times\Lambda^\perp$ (see expression (9.2)).

The end of this section is devoted to the proofs of the three propositions, inspired by [FK00a, AM11].

Proof of Proposition 9.2. The Calderón-Vaillancourt theorem implies that the operators

$$\mathrm{Op}_h\left(\left(1-\chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right)a\left(x,\xi,\frac{P_\Lambda\xi}{h}\right)\right) = \mathrm{Op}_1\left(\left(1-\chi\left(\frac{|P_\Lambda\xi|}{R}\right)\right)a\left(x,h\xi,P_\Lambda\xi\right)\right)$$

are uniformly bounded as $h\to 0$ and $R\to+\infty$. It follows that the family $W_R^{h,\Lambda}$ is bounded in ${S_\Lambda^1}'$, and thus there exists a subsequence (h, w_h) and a distribution $\tilde{\mu}^\Lambda$ such that

$$\lim_{R\to\infty}\lim_{h\to 0}\left\langle W_R^{h,\Lambda}, a\right\rangle_{{S_\Lambda^1}',S_\Lambda^1} = \left\langle \tilde{\mu}^\Lambda, a\left(x,\xi,\eta\right)\right\rangle_{{S_\Lambda^1}',S_\Lambda^1}.$$

Because of the support properties of the function χ, we notice that $\langle\tilde{\mu}^\Lambda, a\rangle_{{S_\Lambda^1}',S_\Lambda^1} = 0$ as soon as the support of a is compact in the variable η. Hence, there exists a distribution $\nu^\Lambda \in \mathscr{D}'(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)$ such that

$$\langle\tilde{\mu}^\Lambda, a(x,\xi,\eta)\rangle_{{S_\Lambda^1}',S_\Lambda^1} = \left\langle \nu^\Lambda, a_{\mathrm{hom}}\left(x,\xi,\frac{\eta}{|\eta|}\right)\right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^\infty(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}.$$

Next, suppose that $a > 0$ (and that $\sqrt{1-\chi}$ is smooth). Then, using [AM11, Corollary 35], and setting

$$b^R(x,\xi) = \left(\left(1-\chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right)a\left(x,\xi,\frac{P_\Lambda\xi}{h}\right)\right)^{\frac{1}{2}},$$

there exists $C>0$ such that for all $h\le h_0$ and $R\ge 1$, we have

$$\left\|\mathrm{Op}_h\left(\left(1-\chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right)a\left(x,\xi,\frac{P_\Lambda\xi}{h}\right)\right) - \mathrm{Op}_h(b^R)^2\right\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \le \frac{C}{R}.$$

As a consequence, we have,

$$\left\langle W_R^{h,\Lambda}, a\right\rangle_{{S_\Lambda^1}',S_\Lambda^1} \ge \|\mathrm{Op}_h(b^R)w_h\|^2_{L^2(\mathbb{T}^2)} - \frac{C}{R}\|w_h\|^2_{L^2(\mathbb{T}^2)},$$

so that the limit $\left\langle \nu^\Lambda, a_{\mathrm{hom}}\left(x,\xi,\frac{\eta}{|\eta|}\right)\right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^\infty(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}$ is nonnegative. The distribution ν^Λ is nonnegative, and is hence a measure. This concludes the proof of Proposition 9.2. □

Proof of Proposition 9.3. First, the proof of the existence of a subsequence (h, w_h) and the measure ρ_Λ satisfying (9.4) is the analogue of Proposition 5.1 in the context of operator valued measures, viewing the sequence w_h as a bounded sequence of $L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))$. It follows the lines of this result, after the adaptation of the symbolic calculus to operator valued symbols (or more precisely, of [Gér91] in the semiclassical setting).

Second, using the definition (9.2) together with (7.6), we have

$$\left\langle W^h_{R,\Lambda}, a \right\rangle_{S^1_\Lambda{}', S^1_\Lambda} = \left(\operatorname{Op}_h \left(\chi \left(\frac{|P_\Lambda \xi|}{Rh} \right) a \left(x, \xi, \frac{P_\Lambda \xi}{h} \right) \right) w_h, w_h \right)_{L^2(\mathbb{T}^2)}$$
$$= \left(\operatorname{Op}_h^{\Lambda^\perp} \circ \operatorname{Op}_h^{\Lambda} \left(\chi \left(\frac{|\eta|}{Rh} \right) a \left(\tilde{\pi}_\Lambda(\sigma, y, \eta), \frac{\eta}{h} \right) \right) T_\Lambda w_h, T_\Lambda w_h \right)_{L^2(\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda)}.$$

Hence, setting

$$a^h_{R,\Lambda}(\sigma, y, \eta) = \chi \left(\frac{|\eta|}{R} \right) a \left(\tilde{\pi}_\Lambda(\sigma, y, h\eta), \eta \right),$$

we obtain

$$\left\langle W^h_{R,\Lambda}, a \right\rangle_{S^1_\Lambda{}', S^1_\Lambda} = \left(\operatorname{Op}_h^{\Lambda^\perp} \circ \operatorname{Op}_1^{\Lambda} \left(a^h_{R,\Lambda}(\sigma, y, \eta) \right) T_\Lambda w_h, T_\Lambda w_h \right)_{L^2(\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda)}.$$

We also notice that $\operatorname{Op}_1^\Lambda \left(a^h_{R,\Lambda} \right) \in \mathcal{K}(L^2(\mathbb{T}_\Lambda))$, for any $\sigma \in \Lambda^\perp$ since $a^h_{R,\Lambda}$ has compact support with respect to η. Moreover, for any $R > 0$ fixed and $a \in S^1_\Lambda$, the Calderón-Vaillancourt theorem yields

$$\operatorname{Op}_1^\Lambda \left(a^h_{R,\Lambda} \right) = \operatorname{Op}_1^\Lambda \left(a^0_{R,\Lambda} \right) + hB$$

for some $B \in \mathcal{L}(L^2(\mathbb{T}_\Lambda))$, uniformly bounded with respect to h. Using (9.4), this implies that for any $R > 0$ fixed and $a \in S^1_\Lambda$, we have

$$\lim_{h \to 0} \left\langle W^h_{R,\Lambda}, a \right\rangle_{S^1_\Lambda{}', S^1_\Lambda} = \operatorname{tr} \left\{ \int_{T^*\mathbb{T}_{\Lambda^\perp}} \operatorname{Op}_1^\Lambda \left(a^0_{R,\Lambda} \right) \rho_\Lambda(ds, d\sigma) \right\}.$$

Moreover, we have

$$\lim_{R \to +\infty} \operatorname{Op}_1^\Lambda \left(a^0_{R,\Lambda} \right) = \operatorname{Op}_1^\Lambda \left(a^0_{\infty,\Lambda} \right) = \operatorname{Op}_1^\Lambda \left(a(\tilde{\pi}_\Lambda(\sigma, y, 0), \eta) \right),$$

in the strong topology of $\mathscr{C}^\infty_c(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{L}(L^2(\mathbb{T}_\Lambda)))$. This proves (9.5) and concludes the proof of Proposition 9.3. $\square$

Proof of Proposition 9.4. Taking $a \in S^1_\Lambda$, independent of the third variable $\eta \in \langle \Lambda \rangle$ gives

$$\left\langle W^h, a\left(x, \xi\right) \right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}^\infty_c(T^*\mathbb{T}^2)} \to \left\langle \mu|_{\mathbb{T}^2 \times \Lambda^\perp}, a \right\rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}^0_c(T^*\mathbb{T}^2)},$$

together with

$$\left\langle W^{h,\Lambda}_R, a \right\rangle_{S^1_\Lambda{}', S^1_\Lambda} \to \left\langle \nu^\Lambda, a \right\rangle_{\mathcal{M}(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda), \mathscr{C}^0_c(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda)},$$

(according to Proposition 9.2) and

$$\left\langle W^h_{R,\Lambda}, a \right\rangle_{S^1_\Lambda{}', S^1_\Lambda} \to \operatorname{tr} \left\{ \int_{T^*\mathbb{T}_{\Lambda^\perp}} \operatorname{Op}_1^\Lambda \left(a(\tilde{\pi}_\Lambda(\sigma, y, 0)) \right) \rho_\Lambda(ds, d\sigma) \right\} = \operatorname{tr} \left\{ \int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{a \circ \tilde{\pi}_\Lambda}(\sigma) \rho_\Lambda(ds, d\sigma) \right\},$$

(according to Proposition 9.3). Now, using the last three equations together with Equation (9.3) directly gives (9.7).

As both terms in the right hand-side of (9.7) are nonnegative measures and the left-hand side is a compactly supported nonnegative measure, this implies that ν^Λ and ρ_Λ are both compactly supported. $\square$

10 Propagation laws for the two-microlocal measures ν^Λ and ρ_Λ

In this section, we study the propagation properties of ν^Λ and ρ_Λ. The key point here is the use of the cutoff function introduced in Proposition 8.3.

We will use repeatedly the following fact, which follows from Item 2 in Proposition 8.3: if A is a bounded operator on $L^2(\mathbb{T}^2)$, we have

$$(Aw_h, w_h)_{L^2(\mathbb{T}^2)} = \left(A\chi_h^\Lambda w_h, \chi_h^\Lambda w_h\right)_{L^2(\mathbb{T}^2)} + \|A\|_{\mathcal{L}(L^2)}\, o(1). \tag{10.1}$$

To simplify the notation, we shall write $A_{c_0,h}$ for $\chi_h^\Lambda A \chi_h^\Lambda$.

10.1 Propagation of ν^Λ

We define for $(x,\xi,\eta) \in T^*\mathbb{T}^2 \times \langle\Lambda\rangle$ and $\tau \in \mathbb{R}$ the flows

$$\phi_\tau^0(x,\xi,\eta) := (x+\tau\xi, \xi, \eta),$$

generated by the vector field $\xi \cdot \partial_x$ and, for $\eta \neq 0$,

$$\phi_\tau^1(x,\xi,\eta) := \left(x + \tau\frac{\eta}{|\eta|}, \xi, \eta\right),$$

generated by the vector field $\frac{\eta}{|\eta|} \cdot \partial_x$. With these definitions, we have the following propagation laws for the two-microlocal measure ν^Λ.

Proposition 10.1. *The measure ν^Λ is ϕ_τ^0- and ϕ_τ^1-invariant, i.e.*

$$(\phi_\tau^0)_*\nu^\Lambda = \nu^\Lambda \quad \textit{and} \quad (\phi_\tau^1)_*\nu^\Lambda = \nu^\Lambda, \quad \textit{for every } \tau \in \mathbb{R}.$$

The key result here is the additional "transverse propagation law" given by the flow ϕ_τ^1. The measure ν^Λ not only propagates along the geodesic flow ϕ_τ^0, but also along directions transverse to $\Lambda^\perp$.

Proof. Fix $a \in S_\Lambda^1$. The computation done in (6.2) is still valid replacing a by $\left(1-\chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right) a\left(x,\xi,\frac{P_\Lambda\xi}{h}\right)$, since it only uses the fact that $\mathrm{Op}_h\left(\left(1-\chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right) a\left(x,\xi,\frac{P_\Lambda\xi}{h}\right)\right)$ is bounded and that $\|P_b^h w_h\|_{L^2(\mathbb{T}^2)} = o(h)$ and $(bw_h, w_h)_{L^2(\mathbb{T}^2)} = o(1)$. This yields

$$\begin{aligned}\lim_{h\to 0} &\left\langle W_R^{h,\Lambda}, \xi\cdot\partial_x a\right\rangle_{{S_\Lambda^1}', S_\Lambda^1} \\ &= \lim_{h\to 0}\left\langle W^h, \xi\cdot\partial_x\left\{\left(1-\chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right) a\left(x,\xi,\frac{P_\Lambda\xi}{h}\right)\right\}\right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}_c^\infty(T^*\mathbb{T}^2)} = 0,\end{aligned}$$

and hence, in the limit $R\to+\infty$, we obtain

$$\left\langle \nu^\Lambda, \xi\cdot\partial_x a_{\mathrm{hom}}\left(x,\xi,\frac{\eta}{|\eta|}\right)\right\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} = 0.$$

Replacing a_{hom} by $a_{\mathrm{hom}} \circ \phi_\tau^0$ and integrating with respect to the parameter τ gives $(\phi_\tau^0)_*\nu^\Lambda = \nu^\Lambda$, which concludes the first part of the proof.

Second, to prove the ϕ_τ^1-invariance of ν^Λ we compute

$$\left\langle \nu^\Lambda, \frac{\eta}{|\eta|}\cdot\partial_x a_{\mathrm{hom}}\left(x,\xi,\frac{\eta}{|\eta|}\right)\right\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} = \lim_{R\to\infty}\lim_{h\to 0}\left\langle W_R^{h,\Lambda}, \frac{\eta}{|\eta|}\cdot\partial_x a\right\rangle_{{S_\Lambda^1}', S_\Lambda^1}. \tag{10.2}$$

Setting

$$a^R(x,\xi,\eta) = \frac{1}{|\eta|}\left(1-\chi\left(\frac{|\eta|}{R}\right)\right)a(x,\xi,\eta),$$

and

$$A^R := \mathrm{Op}_h\left(a^R\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\right) \tag{10.3}$$

we have the relation

$$\left\langle W_R^{h,\Lambda}, \frac{\eta}{|\eta|}\cdot\partial_x a\right\rangle_{S^{1\prime}_\Lambda, S^1_\Lambda} = -\frac{i}{2}([\Delta_\Lambda, A^R]w_h, w_h)_{L^2(\mathbb{T}^2)}$$

where $\Delta_\Lambda = \partial_y^2$ is the laplacian in the direction Λ.

Lemma 10.2. *For any given $c_0>0$ and $R>0$, we have*

$$([\Delta_\Lambda, A^R]w_h, w_h)_{L^2(\mathbb{T}^2)} = ([\Delta_\Lambda, A^R_{c_0,h}]w_h, w_h)_{L^2(\mathbb{T}^2)} + o(1).$$

We postpone the proof of Lemma 10.2 and first indicate how it allows to prove Proposition 10.1. We now know that

$$\left\langle \nu^\Lambda, \frac{\eta}{|\eta|}\cdot\partial_x a_{\mathrm{hom}}\left(x,\xi,\frac{\eta}{|\eta|}\right)\right\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} = \lim_{R\to\infty}\lim_{h\to 0} -\frac{i}{2}([\Delta_\Lambda, A^R_{c_0,h}]w_h, w_h)_{L^2(\mathbb{T}^2)}.$$

Recall that $a\in S^1_\Lambda$ implies that a has only x-Fourier modes in Λ, i.e. $P_\Lambda\xi\cdot\partial_x a = \xi\cdot\partial_x a$. We have also assumed in this section that b has only x-Fourier modes in Λ. As a consequence, we have

$$\begin{aligned}
-\frac{i}{2}([\Delta_\Lambda, A^R_{c_0,h}]w_h, w_h)_{L^2(\mathbb{T}^2)} &= -\frac{i}{2}([\Delta, A^R_{c_0,h}]w_h, w_h)_{L^2(\mathbb{T}^2)}\\
&= \frac{i}{2h^2}\left(\left[P_0^h, A^R_{c_0,h}\right]w_h, w_h\right)_{L^2(\mathbb{T}^2)}.
\end{aligned} \tag{10.4}$$

Developing the last expression of (10.4), we obtain

$$\begin{aligned}
\frac{i}{2h^2}\left(\left[P_0^h, A^R_{c_0,h}\right]w_h, w_h\right)_{L^2(\mathbb{T}^2)} &= \frac{i}{2h^2}\left(A^R_{c_0,h}w_h, P_b^h w_h\right)_{L^2(\mathbb{T}^2)} - \frac{i}{2h^2}\left(A^R_{c_0,h}P_b^h w_h, w_h\right)_{L^2(\mathbb{T}^2)}\\
&\quad - \frac{1}{2h}\left(A^R_{c_0,h}w_h, bw_h\right)_{L^2(\mathbb{T}^2)} - \frac{1}{2h}\left(A^R_{c_0,h}bw_h, w_h\right)_{L^2(\mathbb{T}^2)}.
\end{aligned} \tag{10.5}$$

Since $A^R_{c_0,h}$ is bounded in $\mathcal{L}(L^2(\mathbb{T}^2))$, its adjoint $A^R_{c_0,h}$ is also bounded so that the first two terms in the last expression vanish in the limit $h\to 0$, using $\|P_b^h w_h\|_{L^2(\mathbb{T}^2)} = o(h^2)$. To estimate the last two terms, we use again the boundedness of A^R and $(A^R)^*$ and write

$$\left|\left(A^R_{c_0,h}w_h, bw_h\right)_{L^2(\mathbb{T}^2)}\right| \le \|A^R\|\,\|\chi_h^\Lambda bw_h\|_{L^2(\mathbb{T}^2)} \le 2c_0 h\,\|A^R\|,$$

according to Item 3 in Proposition 8.3. It follows that

$$\limsup_{h\to 0}\left|\frac{1}{2h}\left(A^R_{c_0,h}w_h, bw_h\right)_{L^2(\mathbb{T}^2)} + \frac{1}{2h}\left(A^R_{c_0,h}bw_h, w_h\right)_{L^2(\mathbb{T}^2)}\right| \le 2c_0\sup\|A^R\|.$$

Coming back to the expression (10.2), we obtain

$$\left|\left\langle \nu^\Lambda, \frac{\eta}{|\eta|}\cdot\partial_x a_{\mathrm{hom}}\left(x,\xi,\frac{\eta}{|\eta|}\right)\right\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}\right| \le 2c_0\sup\|A^R\|$$

and since c_0 was arbitrary,

$$\left\langle \nu^\Lambda, \frac{\eta}{|\eta|}\cdot\partial_x a_{\mathrm{hom}}\left(x,\xi,\frac{\eta}{|\eta|}\right)\right\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} = 0$$

Replacing a_{hom} by $a_{\mathrm{hom}}\circ\phi^1_\tau$ and integrating with respect to the parameter τ gives $(\phi^1_\tau)_*\nu^\Lambda = \nu^\Lambda$, which concludes the proof of Proposition 10.1. □

Proof of Lemma 10.2. We are going to show that

$$([\Delta_\Lambda, A^R_{c_0,h}]w_h, w_h)_{L^2(\mathbb{T}^2)} = ([\Delta_\Lambda, A^R]_{c_0,h}w_h, w_h)_{L^2(\mathbb{T}^2)} + o(1). \tag{10.6}$$

Then, using the fact that $[\Delta_\Lambda, A^R]$ is a bounded operator (its symbol is $\left(1-\chi\left(\frac{|\eta|}{R}\right)\right)\frac{\eta}{|\eta|}\cdot\partial_x a(x,\xi,\eta)$), together with (10.1), this is also $([\Delta_\Lambda, A^R]w_h, w_h)_{L^2(\mathbb{T}^2)} + o(1)$.

To prove (10.6), we develop the difference $[\Delta_\Lambda, A^R_{c_0,h}] - [\Delta_\Lambda, A^R]_{c_0,h}$ as

$$[\Delta_\Lambda, A^R_{c_0,h}] - [\Delta_\Lambda, A^R]_{c_0,h} = \left[\partial_y^2, \chi_h^\Lambda\right] A^R\chi_h^\Lambda + \chi_h^\Lambda A^R \left[\partial_y^2, \chi_h^\Lambda\right]. \tag{10.7}$$

Then, writing

$$\left[\partial_y^2, \chi_h^\Lambda\right] = \partial_y^2\chi_h^\Lambda + 2\partial_y\chi_h^\Lambda\ \partial_y,$$

we have

$$\left(\left[\partial_y^2, \chi_h^\Lambda\right] A^R\chi_h^\Lambda w_h, w_h\right)_{L^2(\mathbb{T}^2)} = \left(A^R\chi_h^\Lambda\, w_h, \partial_y^2\chi_h^\Lambda\, w_h\right)_{L^2(\mathbb{T}^2)} + \left(\partial_y\circ A^R\,\chi_h^\Lambda\, w_h, 2\partial_y\chi_h^\Lambda\, w_h\right)_{L^2(\mathbb{T}^2)}.$$

Recalling that the operator $\partial_y\circ A^R$ is bounded, and using Items 4 and 5 in Proposition 8.3, we obtain

$$\left|\left(\left[\partial_y^2, \chi_h^\Lambda\right] A^R\chi_h^\Lambda w_h, w_h\right)_{L^2(\mathbb{T}^2)}\right| \leq C\|\partial_y^2\chi_h^\Lambda\, w_h\|_{L^2(\mathbb{T}^2)} + +C\|\partial_y\chi_h^\Lambda\, w_h\|_{L^2(\mathbb{T}^2)} = o(1).$$

The last term in (10.7) is handled similarly. This finally implies (10.6) and concludes the proof of Lemma 10.2. □

10.2 Propagation of ρ_Λ

We denote by $(\omega_\Lambda^j, e_\Lambda^j)_{j\in\mathbb{N}}$ the eigenvalues and associated eigenfunctions of the operator $-\Delta_\Lambda = -\partial_y^2$ forming a Hilbert basis of $L^2(\mathbb{T}_\Lambda)$. We shall use the projector onto low frequencies of $-\Delta_\Lambda$, i.e., for any $\omega\in\mathbb{R}_+$, the operator

$$\Pi_\Lambda^\omega := \sum_{\omega_\Lambda^j\leq\omega}(\cdot, e_\Lambda^j)_{L^2(\mathbb{T}_\Lambda)}e_\Lambda^j,$$

which has finite rank.

We have the following propagation laws for the two-microlocal measure ρ_Λ.

Proposition 10.3. *1. For any $K\in\mathscr{C}_c^\infty\left(T^*\mathbb{T}_{\Lambda^\perp};\mathcal{K}(L^2(\mathbb{T}_\Lambda))\right)$, independent of s (i.e. $K(s,\sigma) = K(\sigma)$), and any $\omega>0$, we have*

$$\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}}[\Delta_\Lambda, \Pi_\Lambda^\omega K(\sigma)\Pi_\Lambda^\omega]\ \rho_\Lambda(ds,d\sigma)\right\} = 0.$$

2. Moreover, defining

$$M_\Lambda := \int_{\mathbb{T}_{\Lambda^\perp}\times\Lambda^\perp}\rho_\Lambda(ds,d\sigma)\in\mathcal{L}^1(L^2(\mathbb{T}_\Lambda)),$$

we have

$$[\Delta_\Lambda, M_\Lambda] = 0.$$

Remark that for any $\sigma\in\Lambda^\perp$, the operator

$$[\Delta_\Lambda, \Pi_\Lambda^\omega K(\sigma)\Pi_\Lambda^\omega] = \Pi_\Lambda^\omega[\Delta_\Lambda, K(\sigma)]\Pi_\Lambda^\omega,$$

has finite rank, so the right hand-side of Item 1 is well-defined. Note that the definition of M_Λ has a signification since ρ_Λ has a compact support, according to Proposition 9.4.

The commutation relations of Items 1 and 2 in this proposition correspond to propagation laws at the operator level. They are formulated here in a “derivated form”, which, for Item 2 for instance, is equivalent to

$$e^{i\tau\Delta_\Lambda}M_\Lambda e^{-i\tau\Delta_\Lambda} = M_\Lambda, \quad \text{for all } \tau\in\mathbb{R},$$

in the “integrated form”.

Proof. For $K \in \mathscr{C}_c^\infty(\Lambda^\perp; \mathcal{K}(L^2(\mathbb{T}_\Lambda)))$ (in other words $K \in \mathscr{C}_c^\infty(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{K}(L^2(\mathbb{T}_\Lambda)))$ independent of $s \in \mathbb{T}_{\Lambda^\perp}$), we denote

$$K^\omega(\sigma) := \Pi_\Lambda^\omega K(\sigma) \Pi_\Lambda^\omega$$

and we note that K^ω is also in $\mathscr{C}_c^\infty(\Lambda^\perp; \mathcal{K}(L^2(\mathbb{T}_\Lambda)))$. Hence, we have

$$\operatorname{tr}\left\{ \int_{T^*\mathbb{T}_{\Lambda^\perp}} [\Delta_\Lambda, \Pi_\Lambda^\omega K(\sigma)\Pi_\Lambda^\omega]\ \rho_\Lambda(ds, d\sigma) \right\} = -\lim_{h\to 0} \left([-\Delta_\Lambda, K^\omega(hD_s)]T_\Lambda w_h, T_\Lambda w_h\right)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}$$

To show that this limit vanishes, we proceed as in lines (10.4), (10.5) and in the subsequent calculation, replacing the operator A^R by $K^\omega(hD_s)$.

With the notation $\Delta_\Lambda = \partial_y^2$ and $\Delta_{\Lambda^\perp} = \partial_s^2$, we first note that

$$\left([-\Delta_\Lambda, K^\omega(hD_s)]T_\Lambda w_h, T_\Lambda w_h\right)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} = \left([-\Delta, K^\omega(hD_s)]T_\Lambda w_h, T_\Lambda w_h\right)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))},$$

since $\Delta = \Delta_\Lambda + \Delta_{\Lambda^\perp}$ and since $[\Delta_{\Lambda^\perp}, K^\omega(hD_s)] = 0$. As a matter of fact, $K^\omega(hD_s) = \operatorname{Op}_h^\Lambda(K^\omega(\sigma))$ and $\Delta_{\Lambda^\perp} = -h^{-2}\operatorname{Op}_h^\Lambda(|\sigma|^2)$ are both Fourier multipliers.

The following lemma is proved the same way as Lemma 10.2

Lemma 10.4. *For any given $c_0 > 0$, we have*

$$\left([\Delta_\Lambda, K^\omega(\sigma)]T_\Lambda w_h, T_\Lambda w_h\right)_{L^2(\mathbb{T}^2)} = \left([\Delta_\Lambda, K^\omega_{c_0,h}(hD_s)]T_\Lambda w_h, T_\Lambda w_h\right)_{L^2(\mathbb{T}^2)} + o(1).$$

Here $K^\omega_{c_0,h}(hD_s)$ means $\chi_h^\Lambda K^\omega(hD_s)\chi_h^\Lambda$.

Writing

$$-h^2\Delta = T_\Lambda P_b^h T_\Lambda^* - ihb \circ \pi_\Lambda,$$

we have

$$\begin{aligned}
&\left([-\Delta, K^\omega_{c_0,h}(hD_s)]T_\Lambda w_h, T_\Lambda w_h\right)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} \\
&= \frac{1}{h^2}\left(K^\omega_{c_0,h}(hD_s)T_\Lambda w_h, T_\Lambda P_b^h w_h\right)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} - \frac{1}{h^2}\left(K^\omega_{c_0,h}(hD_s)T_\Lambda P_b^h w_h, T_\Lambda w_h\right)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} \\
&\quad + \frac{i}{h}\left(K^\omega_{c_0,h}(hD_s)T_\Lambda w_h, T_\Lambda(bw_h)\right)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} + \frac{i}{h}\left(K^\omega_{c_0,h}(hD_s)T_\Lambda(bw_h), T_\Lambda w_h\right)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}.
\end{aligned}$$

It follows, as in (10.5), that

$$\limsup_{h\to 0} \left| \left([-\Delta, K^\omega_{c_0,h}(hD_s)]T_\Lambda w_h, T_\Lambda w_h\right)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} \right| \leq 2c_0\|K\|$$

and since c_0 was arbitrary, we can conclude that

$$\lim_{h\to 0}\left([\Delta_\Lambda, K^\omega(\sigma)]T_\Lambda w_h, T_\Lambda w_h\right)_{L^2(\mathbb{T}^2)} = 0,$$

which concludes the proof of Item 1.

Item 1 gives, for all $K \in \mathcal{K}(L^2(\mathbb{T}_\Lambda))$ constant (which is possible since $\rho_\Lambda(ds, d\sigma)$ has compact support),

$$0 = \operatorname{tr}\left\{ \int_{T^*\mathbb{T}_{\Lambda^\perp}} [\Delta_\Lambda, K^\omega]\rho_\Lambda(ds, d\sigma) \right\} = \operatorname{tr}\left\{ [\Delta_\Lambda, K^\omega] \int_{T^*\mathbb{T}_{\Lambda^\perp}} \rho_\Lambda(ds, d\sigma) \right\} = \operatorname{tr}\left\{[\Delta_\Lambda, K^\omega]M_\Lambda\right\}.$$

Using that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $A \in \mathcal{L}^1$ and $B \in \mathcal{L}$ together with the linearity of the trace (see [RS80, Theorem VI.25]), we now obtain, for all $K \in \mathcal{K}(L^2(\mathbb{T}_\Lambda))$, and all $\omega > 0$,

$$0 = \operatorname{tr}\left\{[\Delta_\Lambda, \Pi_\Lambda^\omega K \Pi_\Lambda^\omega]M_\Lambda\right\} = \operatorname{tr}\left\{K\Pi_\Lambda^\omega[\Delta_\Lambda, M_\Lambda]\Pi_\Lambda^\omega\right\}.$$

Consequently, we have for all $\omega > 0$, $\Pi_\Lambda^\omega[\Delta_\Lambda, M_\Lambda]\Pi_\Lambda^\omega = 0$ (see [RS80, Theorem VI.26]). Letting ω go to $+\infty$, this yields $[\Delta_\Lambda, M_\Lambda] = 0$ and concludes the proof of Item 2. ∎

11 The measures ν^Λ and ρ_Λ vanish identically. End of the proof of Theorem 2.6

In this section, we prove that both measures ν^Λ and ρ_Λ vanish when paired with the function $\langle b\rangle_\Lambda$. Then, we deduce that these two measures vanish identically. In turn, this implies that $\mu|_{\mathbb{T}^2\times\Lambda^\perp}=0$, and finally that $\mu=0$, which will conclude the proof of Theorem 2.6.

Proposition 11.1. *We have*

$$\left\langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\right\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=0, \quad \textit{and} \quad \operatorname{tr}\{m_{\langle b\rangle_\Lambda}M_\Lambda\}=0.$$

As a consequence, we prove that ρ_Λ and $\nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}$ vanish.

Proposition 11.2. *We have $\rho_\Lambda=0$ and $\nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}=0$. Hence $\mu|_{\mathbb{T}^2\times\Lambda^\perp}=0$.*

This allows to conclude the proof of Theorem 2.6. Indeed, as a consequence of the decomposition formula of Proposition 9.4, we obtain, for all $\Lambda\in\mathcal{P}$, such that $\operatorname{rk}(\Lambda)=1$, $\mu|_{\mathbb{T}^2\times\Lambda^\perp}=0$. Using the decomposition of the measure μ given in Lemma 7.1 together with Lemma 7.4, this yields $\mu=0$ on $\mathbb{T}^2$. This is in contradiction with $\mu(T^*\mathbb{T}^2)=1$ (Proposition 6.1), and this contradiction proves Theorem 2.6.

We now prove Propositions 11.1 and 11.2

Proof of Proposition 11.1. First, (4.22) implies that $(bv_h,v_h)_{L^2(\mathbb{T}^2)}\to 0$, and hence

$$\langle \mu, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}=0.$$

Then the decomposition given in Lemma 7.1 into a sum of nonnegative measures yields that, for all $\Lambda\in\mathcal{P}$,

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}=0, \tag{11.1}$$

since b is also nonnegative. Lemmata 7.2, 7.3 and 7.4 (see also Remark 7.5), then give

$$\begin{aligned}\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)} &= \left\langle \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}, \langle b\rangle_\Lambda\right\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}\\ &= \langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}=0,\end{aligned} \tag{11.2}$$

where the function $\langle b\rangle_\Lambda$ is also nonnegative. The decomposition formula of Proposition 9.4 into the two-microlocal semiclassical measures then yields

$$\begin{aligned}\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)} &= \left\langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\right\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}\\ &\quad+\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{\langle b\rangle_\Lambda}\rho_\Lambda(ds,d\sigma)\right\}.\end{aligned}$$

Besides, the measure $\nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}$ is nonnegative, hence $\left\langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\right\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}\geq 0$. Similarly, $\rho_\Lambda\in\mathcal{M}_c^+(T^*\mathbb{T}_{\Lambda^\perp};\mathcal{L}^1(\mathbb{T}_\Lambda))$ and the operator $m_{\langle b\rangle_\Lambda}\in\mathcal{L}(L^2(\mathbb{T}_\Lambda))$ is selfadjoint and nonnegative, which gives $\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{\langle b\rangle_\Lambda}\rho_\Lambda(ds,d\sigma)\right\}\geq 0$. Using (11.1) and (11.2), this yields

$$\left\langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\right\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=0,$$

and

$$\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{\langle b\rangle_\Lambda}\rho_\Lambda(ds,d\sigma)\right\}=0.$$

In this expression, the operator $m_{\langle b\rangle_\Lambda}$ does not depend on (s,σ), so that

$$0=\operatorname{tr}\left\{m_{\langle b\rangle_\Lambda}\int_{T^*\mathbb{T}_{\Lambda^\perp}}\rho_\Lambda(ds,d\sigma)\right\}=\operatorname{tr}\{m_{\langle b\rangle_\Lambda}M_\Lambda\},$$

which concludes the proof of Proposition 11.1. $\square$

Proof of Proposition 11.2. Let us first prove that $\rho_\Lambda = 0$. We recall that the operator M_Λ is a selfadjoint nonnegative trace-class operator. Moreover, Proposition 10.3 implies that the operators M_Λ and Δ_Λ commute. As a consequence, there exists a Hilbert basis $(\tilde{e}^j_\Lambda)_{j\in\mathbb{N}}$ of $L^2(\mathbb{T}_\Lambda)$ in which M_Λ and Δ_Λ are simultaneously diagonal, i.e. such that

$$-\Delta_\Lambda \tilde{e}^j_\Lambda = \omega^j_\Lambda \tilde{e}^j_\Lambda, \quad \text{and} \quad M_\Lambda \tilde{e}^j_\Lambda = \gamma^j_\Lambda \tilde{e}^j_\Lambda,$$

where $(\gamma^j_\Lambda)_{j\in\mathbb{N}}$ are the associated eigenvalues of M_Λ. In particular, we have $\gamma^j_\Lambda \geq 0$ for all $j \in \mathbb{N}$ (and $\gamma^j_\Lambda \in \ell^1$). Note that the basis $(\tilde{e}^j_\Lambda)_{j\in\mathbb{N}}$ is not necessarily the same as the basis $(e^j_\Lambda)_{j\in\mathbb{N}}$ introduced in Section 10.2.

Using Proposition 11.1, together with the definition of the trace (see for instance [RS80, Theorem VI.18]) we have

$$0 = \operatorname{tr}\{m_{\langle b\rangle_\Lambda} M_\Lambda\} = \sum_{j\in\mathbb{N}} \left(m_{\langle b\rangle_\Lambda} M_\Lambda \tilde{e}^j_\Lambda, \tilde{e}^j_\Lambda \right)_{L^2(\mathbb{T}_\Lambda)} = \sum_{j\in\mathbb{N}} \gamma^j_\Lambda \left(\langle b\rangle_\Lambda \tilde{e}^j_\Lambda, \tilde{e}^j_\Lambda \right)_{L^2(\mathbb{T}_\Lambda)} .$$

Since all terms in this sum are nonnegative (because both γ^j_Λ and $\langle b\rangle_\Lambda$ are), we deduce that for all $j \in \mathbb{N}$,

$$\gamma^j_\Lambda \left(\langle b\rangle_\Lambda \tilde{e}^j_\Lambda, \tilde{e}^j_\Lambda \right)_{L^2(\mathbb{T}_\Lambda)} = 0.$$

Suppose that $\gamma^j_\Lambda \neq 0$ for some $j \in \mathbb{N}$. Then, $\left(\langle b\rangle_\Lambda \tilde{e}^j_\Lambda, \tilde{e}^j_\Lambda \right)_{L^2(\mathbb{T}_\Lambda)} = 0$ where $\langle b\rangle_\Lambda$ is nonnegative and not identically zero on $\mathbb{T}_\Lambda$. This yields $\tilde{e}^j_\Lambda = 0$ on the nonempty open set $\{\langle b\rangle_\Lambda > 0\}$. Using a unique continuation property for eigenfunctions of the Laplace operator on $\mathbb{T}_\Lambda$, we finally obtain that the eigenfunction $\tilde{e}^j_\Lambda$ vanishes identically on $\mathbb{T}_\Lambda$. This is absurd, and thus we must have $\gamma^j_\Lambda = 0$ for all $j \in \mathbb{N}$, so that $M_\Lambda = 0$. Since $\rho_\Lambda \in \mathcal{M}^+(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{L}^1(\mathbb{T}_\Lambda))$, this directly gives $\rho_\Lambda = 0$.

Next, we prove that $\nu^\Lambda = 0$. This is a consequence of the additional propagation law of ν^Λ with respect to the flow ϕ^1_τ (see Section 10.1). Indeed the torus $\mathbb{T}_\Lambda$ has dimension one, $(\phi^1_\tau)_*\nu^\Lambda = \nu^\Lambda$ (according to Proposition 10.1) and, using Proposition 11.1, ν^Λ vanishes on the (nonempty) set $\{\langle b\rangle_\Lambda > 0\} \times \mathbb{R}^2 \times \mathbb{S}_\Lambda$ (with $\{\langle b\rangle_\Lambda > 0\}$ clearly satisfying GCC on $\mathbb{T}_\Lambda$). Hence, $\nu^\Lambda = 0$.

To conclude the proof of Proposition 11.2, it only remains to use the decomposition formula (9.7) which directly yields $\mu|_{\mathbb{T}^2\times\Lambda^\perp} = 0$. □

12 Proof of Proposition 8.2

In this section, we prove Proposition 8.2. For this, we consider two-microlocal semiclassical measures at the scale h^α. The setting is close to that of [FK05].

We shall see that the concentration rate of the sequence v_h towards the direction $\Lambda^\perp$ is of the form h^α for all $\alpha \leq \frac{3+\delta}{4}$.

First, Lemma 7.3 yields $\mu|_{\mathbb{T}^2\times\Lambda^\perp} = \langle\mu\rangle_\Lambda |_{\mathbb{T}^2\times\Lambda^\perp}$ (see also Remark 7.5), i.e.

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, \langle a\rangle_\Lambda\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} ,$$

and it suffices to characterize the action of $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ on $\Lambda^\perp$-invariant symbols. Recall that, for all $a \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2)$,

$$\langle \mu, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \lim_{h\to 0} \left(\operatorname{Op}_h(a) v_h, v_h\right)_{L^2(\mathbb{T}^2)} .$$

In this section, the assumption $\sqrt{b} \in \mathscr{C}^\infty(\mathbb{T}^2)$ is used in an essential way for the propagation result of Lemma 12.2 below. Like in (9.1) and (9.2), let us define :

$$\left\langle V^{h,\Lambda}_R, a\right\rangle_{S^1_\Lambda{}', S^1_\Lambda} := \left\langle V^h, \left(1 - \chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right) a\left(x, \xi, \frac{P_\Lambda \xi}{h}\right)\right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}^\infty_c(T^*\mathbb{T}^2)} , \tag{12.1}$$

$$\left\langle V^h_{R,\Lambda}, a\right\rangle_{S^1_\Lambda{}', S^1_\Lambda} := \left\langle V^h, \chi\left(\frac{|P_\Lambda\xi|}{Rh}\right) a\left(x, \xi, \frac{P_\Lambda \xi}{h}\right)\right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}^\infty_c(T^*\mathbb{T}^2)} , \tag{12.2}$$

for $a \in S^1_\Lambda$.

We take $R = R(h) = h^{-(1-\alpha)}$ for some $\alpha \in (0,1)$, so that $Rh = h^\alpha$. The proof of Proposition 9.2 applies verbatim and shows the existence of a subsequence (h, v_h) and a nonnegative measure $\nu^\Lambda_\alpha \in \mathcal{M}^+(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda)$ such that, for all $a \in S^1_\Lambda$, we have

$$\lim_{h\to 0} \left\langle V^{h,\Lambda}_{R(h)}, a \right\rangle_{{S^1_\Lambda}', S^1_\Lambda} = \left\langle \nu^\Lambda_\alpha, a_{\mathrm{hom}}\left(x, \xi, \frac{\eta}{|\eta|}\right)\right\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda), \mathscr{C}^0_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}.$$

Proposition 12.1. *Let $R(h) = h^{-(1-\alpha)}$ with $\alpha \leq \frac{3+\delta}{4}$. Then*

$$\nu^\Lambda_\alpha|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})\times\mathbb{S}_\Lambda} = 0$$

The proof of Proposition 12.1 relies on the following propagation result.

Lemma 12.2. *For $\alpha \leq \frac{3+\delta}{4}$ the measure ν^Λ_α is ϕ^0_τ- and ϕ^1_τ-invariant, i.e.*

$$(\phi^0_\tau)_*\nu^\Lambda_\alpha = \nu^\Lambda_\alpha \quad \textit{and} \quad (\phi^1_\tau)_*\nu^\Lambda_\alpha = \nu^\Lambda_\alpha, \quad \textit{for every } \tau \in \mathbb{R}.$$

The proof is very similar to that of Proposition 10.1 but does not use Assumption (2.12).

Proof. The proof of ϕ^0_τ-invariance is strictly identical to what has been done for Proposition 10.1 and thus we focus on the ϕ^1_τ-invariance. Equation (10.5) still holds with $R(h) = h^{-(1-\alpha)}$, now reading

$$\begin{aligned}\left\langle V^{h,\Lambda}_{R(h)}, \frac{\eta}{|\eta|}\cdot\partial_x a\right\rangle_{{S^1_\Lambda}', S^1_\Lambda} &= \frac{i}{2h^2}\left(A^{R(h)}v_h, P^h_b v_h\right)_{L^2(\mathbb{T}^2)} - \frac{i}{2h^2}\left(A^{R(h)}P^h_b v_h, v_h\right)_{L^2(\mathbb{T}^2)} \\ &\quad - \frac{1}{2h}\left(A^{R(h)}v_h, b v_h\right)_{L^2(\mathbb{T}^2)} - \frac{1}{2h}\left(A^{R(h)} b v_h, v_h\right)_{L^2(\mathbb{T}^2)}\end{aligned}$$

where A^R was defined in (10.3). Using $\|P^h_b v_h\|_{L^2(\mathbb{T}^2)} = o(h^{1+\delta})$ together with the boundedness of $A^{R(h)}$, it follows that

$$\lim_{h\to 0}\left\langle V^{h,\Lambda}_{R(h)}, \frac{\eta}{|\eta|}\cdot\partial_x a\right\rangle_{{S^1_\Lambda}', S^1_\Lambda} = \lim_{h\to 0}\Big(-\frac{1}{2h}\big(A^{R(h)}v_h, bv_h\big)_{L^2(\mathbb{T}^2)} - \frac{1}{2h}\big(A^{R(h)}bv_h, v_h\big)_{L^2(\mathbb{T}^2)}\Big).$$

Recall from (4.22) that $\|\sqrt{b}v_h\|_{L^2(\mathbb{T}^2)} = o(h^{\frac{1+\delta}{2}})$. In addition, it follows from standard microlocal calculus that

$$[A^{R(h)}, \sqrt{b}] = \mathcal{O}_{\mathcal{L}(L^2)}(R(h)^{-2}).$$

We can thus write

$$\begin{aligned}\left\langle V^{h,\Lambda}_{R(h)}, \frac{\eta}{|\eta|}\cdot\partial_x a\right\rangle_{{S^1_\Lambda}', S^1_\Lambda} &= o(1) - \frac{1}{h}\big(A^{R(h)}\sqrt{b}v_h, \sqrt{b}v_h\big)_{L^2(\mathbb{T}^2)} + \frac{1}{2h}\big(\sqrt{b}[A^{R(h)}, \sqrt{b}]v_h, v_h\big)_{L^2(\mathbb{T}^2)} \\ &\quad + \frac{1}{2h}\big([\sqrt{b}, A^{R(h)}]\sqrt{b}v_h, v_h\big)_{L^2(\mathbb{T}^2)} \\ &= o(1) + o(R(h)^{-2}h^{\frac{-1+\delta}{2}}) = o(1) + o(h^{\frac{3}{2}+\frac{\delta}{2}-2\alpha}),\end{aligned}$$

which vanishes if we take $\alpha \leq \frac{3+\delta}{4}$. ☐

Proof of Proposition 12.1. To prove Proposition 12.1, we first note that

$$\left\langle \nu^\Lambda_\alpha|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\right\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda), \mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} = 0,$$

since ν^Λ_α is (ϕ^0_τ)-invariant and $\left\langle \nu^\Lambda_\alpha, b\right\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda), \mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} = 0$. Then, the ϕ^1_τ-invariance of ν^Λ_α implies that $\nu^\Lambda_\alpha|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})\times\mathbb{S}_\Lambda}$ vanishes. ☐

Proof of Proposition 8.2. Proposition 12.1 implies that

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \lim_{h\to 0}\left(\mathrm{Op}_h\left(\chi\left(\frac{|P_\Lambda\xi|}{h^\alpha}\right)a(x,\xi)\right)v_h, v_h\right)_{L^2(\mathbb{T}^2)}$$

for all $\alpha \le \frac{3+\delta}{4}$ and $a\in\mathscr{C}^\infty_c(T^*\mathbb{T}^2)$. The same holds if we replace χ by χ^2 :

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \lim_{h\to 0}\left(\mathrm{Op}_h\left(\chi^2\left(\frac{|P_\Lambda\xi|}{h^\alpha}\right)a(x,\xi)\right)v_h, v_h\right)_{L^2(\mathbb{T}^2)}.$$

Since

$$\mathrm{Op}_h\left(\chi^2\left(\frac{|P_\Lambda\xi|}{h^\alpha}\right)a(x,\xi)\right) = \mathrm{Op}_h\left(\chi\left(\frac{|P_\Lambda\xi|}{h^\alpha}\right)\right)\mathrm{Op}_h(a)\,\mathrm{Op}_h\left(\chi\left(\frac{|P_\Lambda\xi|}{h^\alpha}\right)\right) + \mathcal{O}(h^{1-\alpha}), \tag{12.3}$$

we obtain

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \lim_{h\to 0}\left(\mathrm{Op}_h(a)\,\mathrm{Op}_h\left(\chi\left(\frac{|P_\Lambda\xi|}{h^\alpha}\right)\right)v_h, \mathrm{Op}_h\left(\chi\left(\frac{|P_\Lambda\xi|}{h^\alpha}\right)\right)v_h\right)_{L^2(\mathbb{T}^2)},$$

for all $\alpha\le\frac{3+\delta}{4}$ and $a\in\mathscr{C}^\infty_c(T^*\mathbb{T}^2)$. $\square$

13 Proof of Proposition 8.3: existence of the cutoff function

Given a constant $c_0>0$, we define the following subsets of $\mathbb{T}^2$:

$$\mathscr{E}_h = \langle\{b>c_0h\}\rangle_\Lambda, \qquad \mathscr{F}_h = \left\langle \bigcup_{x\in\{b>c_0h\}} B(x,(c_0h)^{2\varepsilon})\right\rangle_\Lambda = \bigcup_{x\in\mathscr{E}_h} B(x,(c_0h)^{2\varepsilon}), \qquad \mathscr{G}_h = \mathscr{F}_h\setminus\mathscr{E}_h,$$

where for $U\subset\mathbb{T}^2$, we denote $\langle U\rangle_\Lambda := \bigcup_{\tau\in\mathbb{R}}\{U+\tau\sigma\}$ for some $\sigma\in\Lambda^\perp\setminus\{0\}$. Remark that $\mathscr{E}_h\subset\mathscr{F}_h$ and that $\mathbb{T}^2 = \mathscr{E}_h\cup\mathscr{G}_h\cup(\mathbb{T}^2\setminus\mathscr{F}_h)$. Note also that the sets $\mathscr{E}_h,\mathscr{F}_h$ are non-empty for h small enough, and that $\mathscr{G}_h$ is non empty (for h small enough) as soon as b vanishes somewhere on $\mathbb{T}^2$ (this condition is assumed here since otherwise, GCC is satisfied).

In this section, we construct the cutoff function χ_h^Λ needed to prove the propagation results of Section 10. In particular, this function will be $\Lambda^\perp$-invariant and will satisfy $\chi_h^\Lambda=0$ on $\mathscr{E}_h$ and $\chi_h^\Lambda=1$ on $\mathbb{T}^2\setminus\mathscr{F}_h$.

The proof of Proposition 8.3 relies on three key lemmata. The first key lemma is a precised version of Proposition 6.1 concerning the localization in $T^*\mathbb{T}^2$ of the semiclassical measure μ. It is an intermediate step towards the propagation result stated in Lemma 13.2.

Lemma 13.1. *For any $\chi\in\mathscr{C}^\infty_c(\mathbb{R})$, such that $\chi=1$ in a neighbourhood of the origin, for all $a\in\mathscr{C}^\infty_c(T^*\mathbb{T}^2)$, and $\gamma\le\frac{3+\delta}{2}$, we have*

$$(\mathrm{Op}_h(a)w_h,w_h)_{L^2(\mathbb{T}^2)} = \left(\mathrm{Op}_h(a)\,\mathrm{Op}_h\left(\chi\left(\frac{|\xi|^2-1}{h^\gamma}\right)\right)w_h,w_h\right)_{L^2(\mathbb{T}^2)} + o(h^{\frac{3+\delta}{2}-\gamma})\|\mathrm{Op}_h(a)\|_{\mathcal{L}(L^2)}, \tag{13.1}$$

For all $a\in\mathscr{C}^\infty_c(T^*\mathbb{T}^2)$ and all $\tau\in\mathbb{R}$,

$$(\mathrm{Op}_h(a\circ\phi_\tau)w_h,w_h)_{L^2(\mathbb{T}^2)} = (\mathrm{Op}_h(a)w_h,w_h)_{L^2(\mathbb{T}^2)} + o(\tau h^{\frac{1+\delta}{2}})\|\mathrm{Op}_h(a\circ\phi_t)\|_{L^\infty(0,\tau;\mathcal{L}(L^2(\mathbb{T}^2)))}$$

In this statement, we used the notation

$$\|\mathrm{Op}_h(a\circ\phi_t)\|_{L^\infty(0,\tau;\mathcal{L}(L^2(\mathbb{T}^2)))} := \sup_{t\in(0,\tau)}\|\mathrm{Op}_h(a\circ\phi_t)\|_{\mathcal{L}(L^2(\mathbb{T}^2))}.$$

In turn, this lemma implies the following transport property.

Lemma 13.2. *Suppose that the coefficients α, ε satisfy*

$$0 < 10\varepsilon \leq \alpha, \quad \textit{and} \quad \alpha + 2\varepsilon \leq 1. \tag{13.2}$$

Then, for any time $\tau \in \mathbb{R}$ uniformly bounded with respect to h, and any h-family of functions $\psi = \psi_h \in \mathscr{C}_c^\infty(\mathbb{T}^2)$ satisfying

$$\|\partial_x^k \psi\|_{L^\infty(\mathbb{T}^2)} \leq C_k h^{-2\varepsilon|k|}, \quad \textit{for all } k \in \mathbb{N}^2, \tag{13.3}$$

we have,

$$\begin{aligned}(\psi(s,y)w_h, w_h)_{L^2(\mathbb{T}^2)} &= (\psi(s+\tau,y)w_h, w_h)_{L^2(\mathbb{T}^2)} + (\psi(s-\tau,y)w_h, w_h)_{L^2(\mathbb{T}^2)} \\ &\quad + \mathcal{O}(h^{\alpha-10\varepsilon}) + \mathcal{O}(h^{1-\alpha-2\varepsilon}) + o(h^{\frac{1+\delta}{2}}),\end{aligned} \tag{13.4}$$

where the coordinates (s,y) are the ones introduced in Section 7.3.

In view of Proposition 8.3, this lemma will allow us to propagate the smallness of the sequence w_h above the set $\{b > c_0 h\}$ to all $\mathscr{E}_h$.

The third key lemma states a property of the damping function b, as a consequence of Assumption 2.12.

Lemma 13.3. *There exists $b_0 = b_0(\varepsilon) > 0$ such that for all $x \in \mathbb{T}^2$ satisfying $0 < b(x) < b_0$ and for all $z \in B(x, b(x)^{2\varepsilon})$, we have $b(z) \geq \frac{b(x)}{2}$.*

With these three lemmata, we are now able to prove Proposition 8.3.

Proof of Proposition 8.3. In the coordinates (s,y) of Section 7.3, we can write

$$\mathscr{E}_h = \mathbb{T}_{\Lambda^\perp} \times E_h, \quad \mathscr{F}_h = \mathbb{T}_{\Lambda^\perp} \times F_h, \quad \text{with } E_h \subset F_h \subset \mathbb{T}_\Lambda.$$

Here, F_h is a union of intervals and has uniformly bounded total length. We can hence cover F_h with $C_1 h^{-2\varepsilon}$ subsets of length of order $(c_0 h)^{2\varepsilon}/2$, overlapping on intervals of length of order $(c_0 h)^{2\varepsilon}/10$. Associated to this covering, we denote by $(\psi_j)_{j \in \{1,\dots,J\}}$, $J = J(h)$, a smooth partition of unity on E_h, satisfying moreover

- $\psi_j \in \mathscr{C}_c^\infty(F_h)$;
- $\sum_{j=1}^J \psi_j(y) = 1$ for $y \in E_h$;
- $\|\partial_y^m \psi_j\|_{L^\infty(\mathbb{T}_\Lambda)} \leq C_m h^{-2\varepsilon m}$, for all $m \in \mathbb{N}$;
- $J = J(h) \leq C h^{-2\varepsilon}$.

Similarly, we cover $\mathbb{T}_{\Lambda^\perp}$ with $C_2 h^{-2\varepsilon}$ subsets of length of order $(c_0 h)^{2\varepsilon}/2$, overlapping on intervals of length of order $(c_0 h)^{2\varepsilon}/10$, and define $(\psi_k)_{k \in \{1,\dots,K\}}$ an associated partition of unity on $\mathbb{T}_{\Lambda^\perp}$ satisfying

- $\psi_k \in \mathscr{C}_c^\infty(\mathbb{T}_{\Lambda^\perp})$;
- $\sum_{k=1}^K \psi_k(s) = 1$ for $s \in \mathbb{T}_{\Lambda^\perp}$;
- $\|\partial_s^m \psi_k\|_{L^\infty(\mathbb{T}_{\Lambda^\perp})} \leq C_m h^{-2\varepsilon m}$, for all $m \in \mathbb{N}$;
- $K = K(h) \leq C h^{-2\varepsilon}$,
- for any $k, k_0 \in \{1, \dots, K\}^2$, there exists τ_k satisfying $|\tau_k| \leq \mathrm{Length}(\mathbb{T}_{\Lambda^\perp}) \leq C$ and $\psi_k(s+\tau_k) = \psi_{k_0}(s)$.

We set

$$\psi_{kj}(s,y) := \psi_k(s)\psi_j(y), \qquad \text{and} \qquad \chi_h^\Lambda(s,y) = 1 - \sum_{j=1}^{J}\sum_{k=1}^{K}\psi_{kj}(s,y) \in \mathscr{C}^\infty(\mathbb{T}^2),$$

which satisfies $\partial_s \chi_h^\Lambda(s,y) = 0$, i.e. χ_h^Λ is $\Lambda^\perp$-invariant, together with

- $\chi_h^\Lambda = 0$ on $\mathscr{E}_h$ and hence $b \le c_0 h$ on $\operatorname{supp}(\chi_h^\Lambda)$;
- $\chi_h^\Lambda = 1$ on $\mathbb{T}^2 \setminus \mathscr{F}_h$;
- $\chi_h^\Lambda \in [0,1]$ on $\mathscr{G}_h$, with $|\partial_y \chi_h^\Lambda| \le Ch^{-2\varepsilon}$ and $|\partial_y^2 \chi_h^\Lambda| \le Ch^{-4\varepsilon}$.

To conclude the proof of Proposition 8.3, it remains to check Item 2 ($\|(1-\chi_h^\Lambda)w_h\|_{L^2(\mathbb{T}^2)} = o(1)$), Item 4 ($\|\partial_y \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$) and Item 5 ($\|\partial_y^2 \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$).

Now, let us fix $j_0 \in \{1,\dots,J\}$. Because of the definition of the set $\mathscr{E}_h$, there exists $k_0 \in \{1,\dots,K\}$ and $x_0 \in \{b > c_0 h\}$ such that $\operatorname{supp}(\psi_{k_0 j_0}) \subset B(x_0,(c_0h)^{2\varepsilon})$. According to Lemma 13.3, we have $B(x_0,(c_0h)^{2\varepsilon}) \subset \{b > \frac{c_0 h}{2}\}$, so that $\operatorname{supp}(\psi_{k_0 j_0}) \subset \{b > \frac{c_0 h}{2}\}$. This yields

$$\frac{c_0 h}{2}(\psi_{k_0 j_0} w_h, w_h)_{L^2(\mathbb{T}^2)} \le (b\psi_{k_0 j_0} w_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta}),$$

and hence $(\psi_{k_0 j_0} w_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^\delta)$. Moreover, for any $k \in \{1,\dots,K\}$, there exists τ_k satisfying $|\tau_k| \le C_2$ with

$$\psi_{k j_0}(s+\tau_k, y) = \psi_{k_0 j_0}(s,y).$$

Hence, using (13.4), we obtain

$$\begin{aligned} o(h^\delta) &= (\psi_{k_0 j_0}(s,y) w_h, w_h)_{L^2(\mathbb{T}^2)} = (\psi_{k j_0}(s+\tau_k,y) w_h, w_h)_{L^2(\mathbb{T}^2)} \\ &= (\psi_{k j_0}(s+2\tau_k,y) w_h, w_h)_{L^2(\mathbb{T}^2)} + (\psi_{k j_0}(s,y) w_h, w_h)_{L^2(\mathbb{T}^2)} \\ &\quad + \mathcal{O}(h^{\alpha-10\varepsilon}) + \mathcal{O}(h^{1-\alpha-2\varepsilon}) + o(h^{\frac{1+\delta}{2}}). \end{aligned} \tag{13.5}$$

Since both terms on the right hand-side are nonnegative, this implies $(\psi_{k j_0}(s,y) w_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^\delta)$ as soon as

$$\begin{cases} \alpha - 10\varepsilon > \delta, \\ 1 - \alpha - 2\varepsilon > \delta, \\ \dfrac{1+\delta}{2} \ge \delta, \end{cases}$$

(which implies (13.2)). From now on we will take $\delta = 8\varepsilon$ (this choice is explained in the following lines). The existence of α satisfying this condition together with (8.1) and $\alpha < 3/4$, is equivalent to having $\varepsilon < \frac{1}{76}$.

To conclude the proof of Proposition 8.3, we first compute

$$((1-\chi_h^\Lambda)w_h, w_h)_{L^2(\mathbb{T}^2)} = \sum_{j=1}^{J}\sum_{k=1}^{K}(\psi_{kj} w_h, w_h)_{L^2(\mathbb{T}^2)} = Ch^{-4\varepsilon} o(h^\delta) = o(1),$$

since $\delta \ge 4\varepsilon$. This proves Item 2. Next, we have by construction $\operatorname{supp}(\partial_y^2 \chi_h^\Lambda) \subset \operatorname{supp}(\partial_y \chi_h^\Lambda) \subset \mathscr{G}_h$ with $\|\partial_y \chi_h^\Lambda\|_{L^\infty(\mathbb{T}^2)} = \mathcal{O}(h^{-2\varepsilon})$, $\|\partial_y^2 \chi_h^\Lambda\|_{L^\infty(\mathbb{T}^2)} = \mathcal{O}(h^{-4\varepsilon})$. Hence, covering $\operatorname{supp}(\partial_y \chi_h^\Lambda))$ by balls of radius $(c_0 h)^{2\varepsilon}$ and using a propagation argument similar to (13.5) shows that we have $\|w_h\|_{L^2(\operatorname{supp}(\partial_y \chi_h^\Lambda))} = o(h^{\frac{\delta}{2}})$. We thus obtain

$$\|\partial_y \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(h^{\frac{\delta}{2}-2\varepsilon}) = o(1), \qquad \|\partial_y^2 \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(h^{\frac{\delta}{2}-4\varepsilon}) = o(1),$$

(since $\delta \ge 8\varepsilon$) which concludes the proof of Items 4 and 5, and that of Proposition 8.3. $\square$

To conclude this section, it remains to prove Lemmata 13.2, 13.1 and 13.3. In the following proofs, we shall systematically write η in place of $P_\Lambda \xi$ and σ in place of $(1-P_\Lambda)\xi$ to lighten the notation. Hence, $\xi \in \mathbb{R}^2$ is decomposed as $\xi = \eta + \sigma$ with $\eta \in \langle \Lambda \rangle$ and $\sigma \in \Lambda^\perp$, in accordance to Section 7.3.

Proof of Lemma 13.2 from Lemma 13.1. First, given a function $\psi \in \mathscr{C}_c^\infty(\mathbb{T}^2)$ satisfying (13.3), we have,

$$\begin{aligned}(\psi w_h, w_h)_{L^2(\mathbb{T}^2)} &= (\operatorname{Op}_h(\psi \circ \phi_\tau) w_h, w_h)_{L^2(\mathbb{T}^2)} + o(\tau h^{\frac{1+\delta}{2}}) \| \operatorname{Op}_h(\psi \circ \phi_t)\|_{L^\infty(0,\tau;\mathcal{L}(L^2))} \\ &= \left(\operatorname{Op}_h(\psi \circ \phi_\tau) \operatorname{Op}_h \left(\chi\left(\frac{|\xi|^2 - 1}{h^\gamma}\right)\right) \operatorname{Op}_h \left(\chi \left(\frac{\eta}{2h^\alpha}\right)\right) w_h, w_h \right)_{L^2(\mathbb{T}^2)} \\ &\quad + \left(o(\tau h^{\frac{1+\delta}{2}}) + o(\tau h^{\frac{3+\delta}{2} - \gamma})\right) \| \operatorname{Op}_h(\psi \circ \phi_t)\|_{L^\infty(0,\tau;\mathcal{L}(L^2))},\end{aligned}$$

when using Lemma 13.1 together with $\operatorname{Op}_h\left(\chi\left(\frac{\eta}{2h^\alpha}\right)\right) w_h = w_h$. Next, the pseudodifferential calculus yields

$$\begin{aligned}(\psi w_h, w_h)_{L^2(\mathbb{T}^2)} = &\left(\operatorname{Op}_h \left(\psi \circ \phi_\tau \ \chi\left(\frac{|\xi|^2 - 1}{h^\gamma}\right) \chi \left(\frac{\eta}{2h^\alpha}\right)\right) w_h, w_h \right)_{L^2(\mathbb{T}^2)} + \mathcal{O}(h^{2-\gamma-2\varepsilon}) + \mathcal{O}(h^{1-\alpha-2\varepsilon}) \\ &+ \left(o(\tau h^{\frac{1+\delta}{2}}) + o(\tau h^{\frac{3+\delta}{2} - \gamma})\right) \| \operatorname{Op}_h(\psi \circ \phi_t)\|_{L^\infty(0,\tau;\mathcal{L}(L^2))}. \end{aligned} \tag{13.6}$$

A particular feature of the Weyl quantization in the Euclidean setting is that the Egorov theorem provides an exact formula (see for instance [DS99]): $\operatorname{Op}_h(\psi \circ \phi_t) = e^{-ith\frac{\Delta}{2}} \operatorname{Op}_h(\psi) e^{ith\frac{\Delta}{2}}$, so that $\| \operatorname{Op}_h(\psi \circ \phi_t)\|_{L^\infty(0,\tau;\mathcal{L}(L^2))} \leq C_0$ uniformly with respect to h. Now, remark that the cutoff function $\chi\left(\frac{\eta}{2h^\alpha}\right) \chi\left(\frac{|\xi|^2-1}{h^\gamma}\right)$ can be decomposed (for h small enough) as

$$\chi\left(\frac{\eta}{2h^\alpha}\right) \chi\left(\frac{|\xi|^2-1}{h^\gamma}\right) = \chi\left(\frac{\eta}{2h^\alpha}\right) \left(\tilde{\chi}_\eta^h(\sigma) + \tilde{\chi}_\eta^h(-\sigma)\right)$$

for some nonnegative function $\tilde{\chi}_\eta^h$ such that $(\sigma, \eta) \mapsto \tilde{\chi}_\eta^h(\sigma) \in \mathscr{C}_c^\infty(\mathbb{R}^2)$, such that $\tilde{\chi}_\eta^h(\sigma) = \chi\left(\frac{|\xi|^2-1}{h^\gamma}\right)$ for $\eta \in \operatorname{supp} \chi\left(\frac{\cdot}{2h^\alpha}\right)$ and $\sigma > 0$, and $\tilde{\chi}_\eta^h(\sigma) = 0$ for $\eta \notin \operatorname{supp}\chi\left(\frac{\cdot}{2h^\alpha}\right)$ or $\sigma \leq 0$.

Choosing $\gamma = \alpha$, we have in particular

$$|\sigma - 1| \leq C h^\alpha \quad \text{on } \operatorname{supp}\left(\chi\left(\frac{\eta}{2h^\alpha}\right) \tilde{\chi}_\eta^h(\sigma)\right).$$

Next, we recall that $\psi \circ \phi_\tau(s, y, \sigma, \eta) = \psi(s + \tau\sigma, y + \tau\eta)$, and we focus on the first term (corresponding to $\sigma > 0$) in the right-hand side of the identity

$$\chi\left(\frac{|\xi|^2-1}{h^\alpha}\right) \chi\left(\frac{\eta}{2h^\alpha}\right) \psi \circ \phi_\tau = \chi\left(\frac{\eta}{2h^\alpha}\right) \left(\tilde{\chi}_\eta^h(\sigma) + \tilde{\chi}_\eta^h(-\sigma)\right) \psi \circ \phi_\tau. \tag{13.7}$$

We set

$$\zeta_\tau^{(1)}(s, y, \sigma, \eta) = \chi\left(\frac{\eta}{2h^\alpha}\right) \tilde{\chi}_\eta^h(\sigma) \psi(s + \tau\sigma, y + \tau\eta), \quad \text{and} \quad \zeta_\tau^{(2)}(s, y, \sigma, \eta) = \chi\left(\frac{\eta}{2h^\alpha}\right) \tilde{\chi}_\eta^h(\sigma) \psi(s + \tau, y),$$

and we want to compare $\operatorname{Op}_h(\zeta_\tau^{(1)})$ and $\operatorname{Op}_h(\zeta_\tau^{(2)})$. For this, let us estimate, for multiindices $\ell, m \in \mathbb{N}^2$,

$$\begin{aligned}&\left| \partial_{(s,y)}^\ell \partial_{(\sigma,\eta)}^m \left(\zeta_\tau^{(2)} - \zeta_\tau^{(1)}\right)(s, y, \sigma, \eta)\right| \\ &\qquad \leq C_m \sum_{\nu \leq m} \left| \partial_{(\sigma,\eta)}^{m-\nu} \left(\chi\left(\frac{\eta}{2h^\alpha}\right) \tilde{\chi}_\eta^h(\sigma)\right) \partial_{(s,y)}^\ell \partial_{(\sigma,\eta)}^\nu \left(\psi(s + \tau\sigma, y + \tau\eta) - \psi(s + \tau, y)\right)\right|. \end{aligned} \tag{13.8}$$

On the one hand, we have

$$\left| \partial_{(\sigma,\eta)}^{m-\nu} \left(\chi\left(\frac{\eta}{2h^\alpha}\right) \tilde{\chi}_\eta^h(\sigma)\right)\right| \leq C_{m,\nu} h^{-\alpha|m-\nu|}. \tag{13.9}$$

On the other hand, for $|\nu|>0$ we can also write

$$\begin{aligned}\left|\partial^\ell_{(s,y)}\partial^\nu_{(\sigma,\eta)}\left(\psi(s+\tau\sigma,y+\tau\eta)-\psi(s+\tau,y)\right)\right| &= \left|\partial^\ell_{(s,y)}\partial^\nu_{(\sigma,\eta)}\psi(s+\tau\sigma,y+\tau\eta)\right|\\ &\le C_{\ell,\nu}|\tau|^{|\nu|}h^{-2\varepsilon(|\ell|+|\nu|)}\le C_{\ell,\nu}h^{-2\varepsilon(|\ell|+|\nu|)},\end{aligned}$$

since $|\tau|\le C$.

Finally, for $|\nu|=0$, we apply the mean value theorem to the function

$$(\sigma,\eta)\mapsto \partial^\ell_{(s,y)}\psi(s+\tau\sigma,y+\tau\eta)$$

and write

$$\begin{aligned}&\left|\partial^\ell_{(s,y)}\left(\psi(s+\tau\sigma,y+\tau\eta)-\psi(s+\tau,y)\right)\right|\\ &\qquad\le (|\eta|+|\sigma-1|)\sup_{T^*\mathbb{T}^2}\left|\nabla_{(\sigma,\eta)}\partial^\ell_{(s,y)}\left(\psi(s+\tau\sigma,y+\tau\eta)\right)\right|.\end{aligned}$$

With (13.3), this yields

$$\begin{aligned}\left|\partial^\ell_{(s,y)}\left(\psi(s+\tau\sigma,y+\tau\eta)-\psi(s+\tau,y)\right)\right| &\le (|\eta|+|\sigma-1|)C_\ell h^{-2\varepsilon|\ell|}|\tau|h^{-2\varepsilon}\\ &\le (|\eta|+|\sigma-1|)C_\ell h^{-2\varepsilon(|\ell|+1)},\end{aligned}\tag{13.10}$$

for $|\tau|\le C$.

Using now that $|\eta|\le Ch^\alpha$ and $|\sigma-1|\le Ch^\alpha$ on $\operatorname{supp}\left(\chi\left(\frac{\eta}{2h^\alpha}\right)\tilde\chi^h_\eta(\sigma)\right)$, and combining (13.8), (13.9) and (13.10), we obtain, for all $m\in\mathbb{N}^2,\ell\in\mathbb{N}^2$ and $0<h\le h_0$ sufficiently small,

$$\begin{aligned}h^{|m|}\left|\partial^\ell_{(s,y)}\partial^m_{(\sigma,\eta)}\left(\zeta^{(2)}_\tau-\zeta^{(1)}_\tau\right)(s,y,\sigma,\eta)\right| &\le C_{\ell,m}h^{\alpha-2\varepsilon(|\ell|+1)}h^{|m|}h^{-\alpha|m|}\\ &\quad+C_{\ell,m}\sum_{0<\nu\le m}h^{|m|}h^{-2\varepsilon(|\ell|+|\nu|)}h^{-\alpha|m-\nu|}\\ &\le C_{\ell,m}\left(h^{(1-\alpha)|m|}h^{\alpha-2\varepsilon(|\ell|+1)}+|m|h^{|m|(1-\alpha)}h^{-2\varepsilon|\ell|}h^{\alpha-2\varepsilon}\right)\\ &\le C_{\ell,m}h^{\alpha-2\varepsilon(|\ell|+1)}.\end{aligned}$$

Using a precised version of the Calderón-Vaillancourt theorem, as presented in Theorem A.1 below (in which only $|\ell|=4$ derivations are needed with respect to x in dimension two), we obtain

$$\mathrm{Op}_h(\zeta^{(2)}_\tau)=\mathrm{Op}_h(\zeta^{(1)}_\tau)+\mathcal{O}_{\mathcal{L}(L^2)}(h^{\alpha-10\varepsilon}).$$

Similarly, we have

$$\mathrm{Op}_h\left(\chi\left(\frac{\eta}{2h^\alpha}\right)\tilde\chi^h_\eta(-\sigma)\psi(s+\tau\sigma,y+\tau\eta)\right)=\mathrm{Op}_h\left(\chi\left(\frac{\eta}{2h^\alpha}\right)\tilde\chi^h_\eta(-\sigma)\psi(s-\tau,y)\right)+\mathcal{O}_{\mathcal{L}(L^2)}(h^{\alpha-10\varepsilon}).$$

Coming back to (13.6) and using (13.7), we finally obtain, for all $|\tau|\le C$,

$$\begin{aligned}(\psi w_h,w_h)_{L^2(\mathbb{T}^2)} &= \left(\mathrm{Op}_h\left(\chi\left(\frac{\eta}{2h^\alpha}\right)\tilde\chi^h_\eta(\sigma)\psi(s+\tau,y)\right)w_h,w_h\right)_{L^2(\mathbb{T}^2)}\\ &\quad+\left(\mathrm{Op}_h\left(\chi\left(\frac{\eta}{2h^\alpha}\right)\tilde\chi^h_\eta(-\sigma)\psi(s-\tau,y)\right)w_h,w_h\right)_{L^2(\mathbb{T}^2)}\\ &\quad+\mathcal{O}(h^{\alpha-10\varepsilon})+\mathcal{O}(h^{1-\alpha-2\varepsilon})+o(h^{\frac{1+\delta}{2}})+o(h^{\frac{3+\delta}{2}-\alpha}).\end{aligned}$$

With the pseudodifferential calculus, this yields (13.4), which concludes the proof of Lemma 13.2. □

Proof of Lemma 13.1. Here, we only have to make more precise some arguments in the proof of Lemma 6.1. Recall that according to Lemma 8.1, w_h satisfies $P^h_b w_h=o(h^{2+\delta})$.

First, we take $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$, such that $\chi = 1$ in a neighbourhood of the origin. Hence, $\frac{1-\chi(r)}{r} \in \mathscr{C}^\infty(\mathbb{R})$ and we have the exact composition formula

$$\mathrm{Op}_h\left(1 - \chi\left(\frac{|\xi|^2 - 1}{h^\gamma}\right)\right) = \mathrm{Op}_h\left(\left(1 - \chi\left(\frac{|\xi|^2 - 1}{h^\gamma}\right)\right)\frac{h^\gamma}{|\xi|^2 - 1}\right)\frac{P_0^h}{h^\gamma},$$

since both operators are Fourier multipliers. Moreover, $\mathrm{Op}_h\left(\left(1 - \chi\left(\frac{|\xi|^2-1}{h^\gamma}\right)\right)\frac{h^\gamma}{|\xi|^2-1}\right)$ is uniformly bounded as an operator of $\mathcal{L}(L^2(\mathbb{T}^2))$. As a consequence, we have

$$\begin{aligned}
&\left(\mathrm{Op}_h(a)\,\mathrm{Op}_h\left(1 - \chi\left(\frac{|\xi|^2 - 1}{h^\gamma}\right)\right)w_h, w_h\right)_{L^2(\mathbb{T}^2)} \\
&= \left(\mathrm{Op}_h(a)\,\mathrm{Op}_h\left(\left(1 - \chi\left(\frac{|\xi|^2 - 1}{h^\gamma}\right)\right)\frac{h^\gamma}{|\xi|^2 - 1}\right)\frac{P_0^h}{h^\gamma}w_h, w_h\right)_{L^2(\mathbb{T}^2)} \\
&= \left(A\frac{P_b^h}{h^\gamma}w_h, w_h\right)_{L^2(\mathbb{T}^2)} - \left(A\frac{ihb}{h^\gamma}w_h, w_h\right)_{L^2(\mathbb{T}^2)},
\end{aligned}$$

where $A = \mathrm{Op}_h(a)\,\mathrm{Op}_h\left(\left(1 - \chi\left(\frac{|\xi|^2-1}{h^\gamma}\right)\right)\frac{h^\gamma}{|\xi|^2-1}\right)$ is bounded on $L^2(\mathbb{T}^2)$. Using $P_b^h w_h = o(h^{2+\delta})$ and $(bw_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta})$, this gives

$$\left(\mathrm{Op}_h(a)\,\mathrm{Op}_h\left(1 - \chi\left(\frac{|\xi|^2 - 1}{h^\gamma}\right)\right)w_h, w_h\right)_{L^2(\mathbb{T}^2)} = o(h^{\frac{3+\delta}{2}-\gamma})\|\mathrm{Op}_h(a)\|_{\mathcal{L}(L^2)},$$

which in turn implies (13.1).

Next, Identity (6.2) yields, for all $a \in \mathscr{C}_c^\infty(\mathbb{T}^2)$,

$$\begin{aligned}
(\mathrm{Op}_h(\xi \cdot \partial_x a)w_h, w_h)_{L^2(\mathbb{T}^2)} &= \frac{i}{2h}\left(\mathrm{Op}_h(a)w_h, P_b^h w_h\right)_{L^2(\mathbb{T}^2)} - \frac{i}{2h}\left(\mathrm{Op}_h(a)P_b^h w_h, w_h\right)_{L^2(\mathbb{T}^2)} \\
&\quad - \frac{1}{2}\left(\mathrm{Op}_h(a)w_h, bw_h\right)_{L^2(\mathbb{T}^2)} - \frac{1}{2}\left(\mathrm{Op}_h(a)bw_h, w_h\right)_{L^2(\mathbb{T}^2)} \\
&= o(h^{1+\delta})\|\mathrm{Op}_h(a)\|_{\mathcal{L}(L^2)} + o(h^{\frac{1+\delta}{2}})\|\mathrm{Op}_h(a)\|_{\mathcal{L}(L^2)},
\end{aligned}$$

as a consequence of $P_b^h w_h = o(h^{2+\delta})$ and $(bw_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta})$. Applying this identity to $a \circ \phi_t$ in place of a, and integrating on $t \in [0, \tau]$ finally gives

$$(\mathrm{Op}_h(a \circ \phi_\tau)w_h, w_h)_{L^2(\mathbb{T}^2)} = (\mathrm{Op}_h(a)w_h, w_h)_{L^2(\mathbb{T}^2)} + o(\tau h^{\frac{1+\delta}{2}})\|\mathrm{Op}_h(a \circ \phi_t)\|_{L^\infty(0,\tau;\mathcal{L}(L^2))},$$

which concludes the proof of Lemma 13.1. $\square$

Proof of Lemma 13.3. Here, $B := B(x, b(x)^{2\varepsilon})$ denotes the euclidian ball in $\mathbb{T}^2$ centered at x of radius $b(x)^{2\varepsilon}$. Setting

$$M := \sup_{z \in B} b(z), \qquad m := \inf_{z \in B} b(z),$$

we have

$$|\nabla b(z)| \leq C_\varepsilon b^{1-\varepsilon}(z) \leq C_\varepsilon M^{1-\varepsilon}, \quad \text{for all } z \in B,$$

as a consequence of Assumption 2.12. Moreover, the mean value theorem yields

$$b(x) - C_\varepsilon M^{1-\varepsilon} b(x)^{2\varepsilon} \leq b(z) \leq b(x) + C_\varepsilon M^{1-\varepsilon} b(x)^{2\varepsilon}, \quad \text{for } z \in B,$$

and, in particular,

$$m \geq b(x) - C_\varepsilon M^{1-\varepsilon} b(x)^{2\varepsilon}, \quad \text{and} \quad M \leq b(x) + C_\varepsilon M^{1-\varepsilon} b(x)^{2\varepsilon}. \tag{13.11}$$

Now, defining $f(M) := b(x) + C_\varepsilon M^{1-\varepsilon} b(x)^{2\varepsilon}$, we see that f is a strictly concave function with $f(0) = b(x) > 0$. There exists a unique $M_0 \in \mathbb{R}_+$ satisfying $f(M_0) = M_0$. Moreover, we have $M \leq f(M)$ if and only if $M \leq M_0$. Taking b_0 sufficiently small so that $b_0 + C_\varepsilon b_0^{1+\varepsilon^2} \leq b_0^{1-\varepsilon}$, we obtain $f(b(x)^{1-\varepsilon}) \leq b(x)^{1-\varepsilon}$. In particular, this gives $M_0 \leq b(x)^{1-\varepsilon}$ and hence $M \leq b(x)^{1-\varepsilon}$ according to the second estimate of (13.11). Coming back to the first estimate of (13.11), this yields

$$m \geq b(x) - C_\varepsilon b(x)^{(1-\varepsilon)^2} b(x)^{2\varepsilon} = b(x) - C_\varepsilon b(x)^{1+\varepsilon^2}.$$

Taking b_0 sufficiently small so that $b_0 - C_\varepsilon b_0^{1+\varepsilon^2} \geq \frac{b_0}{2}$, we obtain $m \geq \frac{b(x)}{2}$, which concludes the proof of Lemma 13.3. ☐

Part IV

An *a priori* lower bound for decay rates on the torus: proof of Theorem 2.5

Under the assumption

$$\overline{\{b > 0\}} \cap \{x_0 + \tau\xi_0, \tau \in \mathbb{R}\} = \emptyset, \tag{13.12}$$

for some $(x_0, \xi_0) \in T^*\mathbb{T}^2$, $\xi_0 \neq 0$, we construct in this section a constant $\kappa_0 > 0$ and a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathcal{O}(1)$-quasimodes in the limit $n \to +\infty$ for the family of operators $P(in\kappa_0)$.

We use the notation introduced in Sections 7.1 and 9. First, note that, as a consequence of (13.12), ξ_0 is necessarily a rational direction, and the set $\{x_0 + \tau\xi_0, \tau \in \mathbb{R}\}$ is a one-dimensional subtorus of $\mathbb{T}^2$, given by

$$\{x_0 + \tau\xi_0, \tau \in \mathbb{R}\} = \overline{\{x_0 + \tau\xi_0, \tau \in \mathbb{R}\}} = x_0 + \mathbb{T}_{\Lambda_{\xi_0}^\perp}, \quad \text{with } \Lambda_{\xi_0} \in \mathcal{P}.$$

Let $\chi \in \mathscr{C}_c^\infty(\mathbb{T}^2)$ such that χ has only x-Fourier modes in Λ_{ξ_0}, $\chi = 0$ on a neighbourhood of $\overline{\{b > 0\}}$ and $\chi = 1$ on $x_0 + \mathbb{T}_{\Lambda_{\xi_0}^\perp}$.

From Assumption (13.12), we have $\mathrm{rk}(\Lambda_{\xi_0}) = 1$, so that one can find $k \in \Lambda_{\xi_0}^\perp \cap \mathbb{Z}^2 \setminus \{0\}$. Besides, for all $n \in \mathbb{N}$ we have $nk \in \Lambda_{\xi_0}^\perp \cap \mathbb{Z}^2 \setminus \{0\}$.

We then define the sequence of *quasimodes* $(\varphi_n)_{n\in\mathbb{N}}$ by

$$\varphi_n(x) = \chi(x) e^{ink\cdot x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{T}^2.$$

We have $\varphi_n \in \mathscr{C}^\infty(\mathbb{T}^2)$, together with the decoupling

$$\varphi_n \circ \pi_{\Lambda_{\xi_0}}(s, y) = \chi(y) e^{ink\cdot s}, \quad n \in \mathbb{N}, \quad (s, y) \in \mathbb{T}_{\Lambda_{\xi_0}^\perp} \times \mathbb{T}_{\Lambda_{\xi_0}}.$$

This yields

$$\begin{aligned} -\big(T_{\Lambda_{\xi_0}} \Delta T_{\Lambda_{\xi_0}}^*\big)\varphi_n \circ \pi_{\Lambda_{\xi_0}}(s, y) &= -\big(\Delta_{\Lambda_{\xi_0}} + \Delta_{\Lambda_{\xi_0}^\perp}\big)\varphi_n \circ \pi_{\Lambda_{\xi_0}}(s, y) \\ &= -e^{ink\cdot s} \Delta_{\Lambda_{\xi_0}} \chi(y) + n^2|k|^2 \chi(y) e^{ink\cdot s}. \end{aligned}$$

Moerover, $b\varphi_n = 0$, according to their respective supports. Hence, recalling that $P(in|k|) = -\Delta - n^2|k|^2 + in|k|b(x)$, we have

$$\big(T_{\Lambda_{\xi_0}} P(in|k|) T_{\Lambda_{\xi_0}}^*\big)\varphi_n \circ \pi_{\Lambda_{\xi_0}} = -e^{ink\cdot s} \Delta_{\Lambda_{\xi_0}} \chi(y),$$

and

$$\|P(in|k|)\varphi_n\|_{L^2(\mathbb{T}^2)} = \|\big(T_{\Lambda_{\xi_0}} P(in|k|) T_{\Lambda_{\xi_0}}^*\big)\varphi_n \circ \pi_{\Lambda_{\xi_0}}\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}^\perp} \times \mathbb{T}_{\Lambda_{\xi_0}})} = C_0 \|\Delta_{\Lambda_{\xi_0}} \chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}.$$

Since we also have $\|\varphi_n\|_{L^2(\mathbb{T}^2)} = \|T_{\Lambda_{\xi_0}}\varphi_n\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}^\perp}\times\mathbb{T}_{\Lambda_{\xi_0}})} = C_0\|\chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}$, we obtain, for all $n \in \mathbb{N}$,

$$\|P^{-1}(in|k|)\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \geq \frac{\|\varphi_n\|_{L^2(\mathbb{T}^2)}}{\|P(in|k|)\varphi_n\|_{L^2(\mathbb{T}^2)}} = \frac{\|\chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}}{\|\Delta_{\Lambda_{\xi_0}}\chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}} = C > 0,$$

which concludes the proof of Theorem 2.5. □

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A Pseudodifferential calculus

In the main part of the article, we use the semiclassical Weyl quantization, that associates to a function a on $T^*\mathbb{R}^2$ an operator $\mathrm{Op}_h(a)$ defined by

$$\big(\mathrm{Op}_h(a)u\big)(x) := \frac{1}{(2\pi h)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\frac{i}{h}\xi\cdot(x-y)} a\left(\frac{x+y}{2},\xi\right) u(y)dy\, d\xi. \tag{A.1}$$

For smooth functions a with uniformly bounded derivatives, $\mathrm{Op}_h(a)$ defines a continuous operator on $\mathscr{S}(\mathbb{R}^2)$, and also by duality on $\mathscr{S}'(\mathbb{R}^2)$. On a manifold, the quantization Op_h may be defined by working in local coordinates with a partition of unity. On the torus, formula (A.1) still makes sense : taking $a \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ is equivalent to taking $a \in \mathscr{C}^\infty(\mathbb{R}^2\times\mathbb{R}^2)$, $(2\pi\mathbb{Z})^2$-periodic with respect to the x-variable. Then the operator defined by (A.1) preserves the space of $(2\pi\mathbb{Z})^2$-periodic distributions on $\mathbb{R}^2$, and hence $\mathscr{D}'(\mathbb{T}^2)$.

We sometimes write, with $D := \frac{1}{i}\partial$,

$$a(x,hD) = \mathrm{Op}_h(a).$$

We also note that $\mathrm{Op}_1(a)$ is the classical Weyl quantization, and that we have the relation

$$a(x,hD) = \mathrm{Op}_h(a(x,\xi)) = \mathrm{Op}_1(a(x,h\xi)).$$

Theorem A.1. *There exists a constant $C > 0$ such that for any $a \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ with uniformly bounded derivatives, we have*

$$\|\mathrm{Op}_1(a)\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \leq C \sum_{\alpha,\beta\in\{0,1,2\}^2} \|\partial_x^\alpha\partial_\xi^\beta a\|_{L^\infty(T^*\mathbb{T}^2)}.$$

Equivalently, this can be rewritten as

$$\|\mathrm{Op}_h(a)\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \leq C \sum_{\alpha,\beta\in\{0,1,2\}^2} h^{|\beta|}\|\partial_x^\alpha\partial_\xi^\beta a\|_{L^\infty(T^*\mathbb{T}^2)}.$$

This precised version of the Calderón-Vaillancourt theorem is needed in Section 13, and proved in [Cor75, Theorem B_ρ] or [CM78, Théorème 3]. Here in dimension two, this means that only $|\alpha| = 4$ derivations are needed with respect to the space variable x.

B Spectrum of $P(z)$ for a piecewise constant damping (by Stéphane Nonnenmacher)

In this Appendix we provide an explicit description of some part of the *spectrum* of the damped wave equation (1.1) on $\mathbb{T}^2$, for a damping function proportional to the characteristic function of a vertical strip. We identify the torus $\mathbb{T}^2$ with the square $\{-1/2 \leq x < 1/2,\ 0 \leq y < 1\}$. We choose some half-width $\sigma \in (0, 1/2)$, and consider a vertical strip of width 2σ. Due to translation symmetry of $\mathbb{T}^2$, we may center this strip on the axis $\{x = 0\}$. Choosing a damping strength $\widetilde{B} > 0$, we then get the damping function

$$b(x, y) = b(x) = \begin{cases} 0, & |x| \leq \sigma, \\ \widetilde{B}, & \sigma < |x| \leq 1/2 \,. \end{cases} \tag{B.1}$$

The reason for centering the strip at $x = 0$ is the parity of the problem w.r.t. that axis, which greatly simplifies the computations.

We are interested in the spectrum of the operator $\mathcal{A}$ generating the equation (1.1), which amounts to solving the eigenvalue problem

$$P(z)u = 0, \quad \text{for} \quad P(z) = -\Delta + zb(x) + z^2, \quad z \in \mathbb{C}\,,\ u \in L^2(\mathbb{T}^2)\,,\ u \not\equiv 0\,. \tag{B.2}$$

This spectrum consists in a discrete set $\{z_j\}$, which is symmetric w.r.t. the horizontal axis: indeed, any solution (z, u) admits a "sister" solution $(\bar{z}, \bar{u})$. Furthermore, any solution with $\operatorname{Im} z \neq 0$ satisfies

$$\operatorname{Re} z = -\frac{1}{2} \frac{(u, bu)_{L^2(\mathbb{T}^2)}}{\|u\|^2_{L^2(\mathbb{T}^2)}}, \quad \text{and thus} \quad -\widetilde{B}/2 \leq \operatorname{Re} z \leq 0\,. \tag{B.3}$$

We may thus restrict ourselves to the half-strip $\{-\widetilde{B}/2 \leq \operatorname{Re} z \leq 0,\ \operatorname{Im} z > 0\}$.

Our aim is to find high frequency eigenvalues ($\operatorname{Im} z \gg 1$) which are as close as possible to the imaginary axis. We will prove the following

Proposition B.1. *There exists $C_0 > 0$ such that the spectrum* (B.3) *for the damping function* (B.1) *contains an infinite subsequence $\{z_i\}$ such that $\operatorname{Im} z_i \to \infty$ and $|\operatorname{Re} z_i| \leq \frac{C_0}{(\operatorname{Im} z_i)^{3/2}}$.*

The proof of the proposition will actually give an explicit value for C_0, as a function of $\tilde{B}$, σ.

Proof. To study the high frequency limit $\operatorname{Im} z \to \infty$ we will change of variables and take

$$z = i(1/h + \widetilde{\zeta})\,,$$

where $h \in (0, 1]$ will be a small parameter, while $\widetilde{\zeta} \in \mathbb{C}$ is assumed to be uniformly bounded when $h \to 0$. The eigenvalue equation then takes the form

$$(-h^2\Delta + ih(1 + h\widetilde{\zeta})b)u = \big(1 + 2h\widetilde{\zeta}(1 + h\widetilde{\zeta}/2)\big)u\,. \tag{B.4}$$

Having chosen b independent of y, we may naturally Fourier transform along this direction, that is look for solutions of the form $u(x, y) = e^{2i\pi ny}v(x)$, $n \in \mathbb{Z}$. For each n, we now have to solve the 1-dimensional problem

$$(-h^2\partial^2/\partial_x^2 + ih(1 + h\widetilde{\zeta})b(x))v = \big(1 - (2\pi hn)^2 + 2h\widetilde{\zeta}(1 + h\widetilde{\zeta}/2)\big)\, v\,. \tag{B.5}$$

Let us call

$$B \overset{\text{def}}{=} \widetilde{B}(1 + h\widetilde{\zeta}), \quad \zeta \overset{\text{def}}{=} \widetilde{\zeta}(1 + h\widetilde{\zeta}/2)\,.$$

In terms of these parameters, the above equation reads:

$$(-h^2\partial^2/\partial_x^2 + ihB\,\mathbb{1}_{\{\sigma<|x|\leq 1/2\}}(x))v = E\,v\,, \qquad \text{with} \quad E = 1 - (2\pi hn)^2 + 2h\zeta\,. \tag{B.6}$$

Since we will assume throughout that $\widetilde{\zeta} = \mathcal{O}(1)$, we will have in the semiclassical limit

$$B = \widetilde{B} + \mathcal{O}(h), \quad \widetilde{\zeta} = \zeta(1 - h\zeta/2 + \mathcal{O}(h^2))\,. \tag{B.7}$$

At leading order we may forget that the variables B, ζ are not independent from one another, and consider (B.6) as a *bona fide* linear eigenvalue problem.

Since the function $b(x)$ is even, we may separately search for even, resp. odd solutions $v(x)$. Let us start with the even solutions. Since $b(x)$ is piecewise constant, any even and periodic solution $v(x)$ takes the following form on $[-1/2, 1/2]$ (up to a global normalization factor):

$$v(x) = \begin{cases} \cos(k\,x), & |x| \le \sigma, \\ \beta \cos\big(k'\,(1/2 - |x|)\big), & \sigma < |x| \le 1/2, \end{cases}, \tag{B.8}$$

$$k = \frac{E^{1/2}}{h}, \quad k' = \frac{(E - ihB)^{1/2}}{h}. \tag{B.9}$$

We notice that k, k' are defined modulo a change of sign, so we may always assume that $\operatorname{Re} k \ge 0$, $\operatorname{Re} k' \ge 0$. The factor β is obtained by imposing the continuity of v and of its derivative v' at the discontinuity point $x = \sigma$ (we use the notation $\sigma' \stackrel{\text{def}}{=} 1/2 - \sigma$):

$$\begin{aligned} \cos(k\sigma) &= \beta \cos(k'\sigma'), \\ -k\,\sin(k\sigma) &= \beta k'\,\sin(k'\sigma'). \end{aligned}$$

The ratio of these two equations provides the quantization condition for the even solutions:

$$\tan(k\sigma) = -\frac{k'}{k}\,\tan(k'\sigma'). \tag{B.10}$$

Similarly, any odd eigenfunction takes the form (modulo a global normalization factor):

$$v(x) = \begin{cases} \sin(k\,x), & |x| \le \sigma, \\ \beta\,\mathrm{sgn}(x)\,\sin(k'(1/2 - |x|)), & \sigma < |x| \le 1/2, \end{cases},$$

so the associated eigenvalues should satisfy the condition

$$\tan(k\sigma) = -\frac{k}{k'}\,\tan(k'\sigma'). \tag{B.11}$$

We will now study the solutions of the quantization conditions (B.10) and (B.11), taking into account the relations (B.9) between the wavevectors k, k' and the energy E. To describe the full spectrum (which we plan to present in a separate publication), we would need to consider several régimes, depending on the relative scales of E and h. However, since we are only interested here in proving Proposition B.1, we will focus on the régime leading to the smallest possible values of $|\operatorname{Im}\widetilde{\zeta}| = |\operatorname{Re} z|$. What characterizes the corresponding eigenmodes $v(x)$? From (B.3) we see that the mass of $v(x)$ in the damped region, $2\int_\sigma^{1/2} |v(x)|^2\,dx$, should be small compared to its full mass. Intuitively, if such a mode were carrying a large horizontal "momentum" $\operatorname{Re}(hk)$ in the undamped region, it would then strongly penetrate the damped region, because the boundary at $x = \sigma$ is not reflecting. As a result, the mass in the damped region would be of the same order of magnitude as the one in the undamped one. This hand-waving argument explains why we choose to investigate the eigenmodes for which hk is the smallest possible, namely of order $\mathcal{O}(h)$. This implies that $E = (hk)^2 = \mathcal{O}(h^2)$, which means that almost all of the energy is carried by the vertical momentum:

$$hn = (2\pi)^{-1} + \mathcal{O}(h).$$

The study of the full spectrum actually confirms that the smallest values of $\operatorname{Im}\widetilde{\zeta}$ are obtained in this régime.

Eq.(B.9) implies that the wavevector k' in the damped region is then much larger than k:

$$k' = \frac{(-ihB + (hk)^2)^{1/2}}{h} = e^{-i\pi/4}(B/h)^{1/2} + \mathcal{O}(h^{1/2}).$$

$\operatorname{Im} k'\sigma' \approx -\sigma'(B/2h)^{1/2}$ is negative and large, so that $\tan(k'\sigma') = -i + \mathcal{O}(e^{2\operatorname{Im}(k'\sigma')})$, uniformly w.r.t. $\operatorname{Re}(k'\sigma')$.

Even eigenmodes

In this situation the even quantization condition (B.10) reads

$$\tan(k\sigma) = i\frac{k'}{k}\left(1 + \mathcal{O}(e^{-\sigma'(2B/h)^{1/2}})\right). \tag{B.12}$$

Since the r.h.s. is large, $k\sigma$ must be close to a pole of the tangent function. Hence, for each integer m in a bounded interval[1] $0 \le m \le M$ we look for a solution of the form

$$k_{m+1/2} = \frac{\pi(m+1/2)}{\sigma} + \delta k_{m+1/2}, \quad \text{with} \quad |\delta k_{m+1/2}| \ll 1\,.$$

The quantization condition (B.12) then reads

$$\begin{aligned}\sigma\delta k_{m+1/2} + \mathcal{O}((\delta k_{m+1/2})^2) &= i\,\frac{k_{m+1/2}}{e^{-i\pi/4}(B/h)^{1/2} + \mathcal{O}(h^{1/2})}\left(1 + \mathcal{O}(e^{-\sigma'(2B/h)^{1/2}})\right)\\ \Longrightarrow k_{m+1/2} &= \frac{\pi(m+1/2)}{\sigma}\Big(1 + h^{1/2}\frac{e^{i3\pi/4}}{\sigma B^{1/2}} + \mathcal{O}(h)\Big)\,.\end{aligned}$$

Using (B.6), the corresponding spectral parameter ζ is then given by

$$\begin{aligned}\zeta_{n,m+1/2} &= \frac{(hk_{m+1/2})^2 + (2\pi hn)^2 - 1}{2h}\\ &= \frac{(2\pi hn)^2 - 1}{2h} + \frac{h}{2}\Big(\frac{\pi(m+1/2)}{\sigma}\Big)^2 + h^{3/2}\left(\frac{\pi(m+1/2)}{\sigma}\right)^2\frac{e^{i3\pi/4}}{\sigma B^{1/2}} + \mathcal{O}(h^2)\,.\end{aligned}$$

From the assumptions on the quantum numbers n, m, we check that $\zeta_{n,m+1/2} = \mathcal{O}(1)$. We may now go back to the original variables $\widetilde{\zeta}$, $\widetilde{B}$, using the relations (B.7). The spectral parameter $\widetilde{\zeta}$ has an imaginary part

$$\operatorname{Im}\widetilde{\zeta}_{n,m+1/2} = \operatorname{Im}\zeta_{n,m+1/2}(1 - h\operatorname{Re}\zeta_{n,m+1/2}) + \mathcal{O}(h^2) = h^{3/2}\frac{(\pi(m+1/2))^2}{\sigma^3(2\widetilde{B})^{1/2}} + \mathcal{O}(h^2)\,. \tag{B.13}$$

Returning back to the spectral variable z, the above expression gives a string of eigenvalues $\{z_{n,m+1/2}\}$ with $\operatorname{Im} z_{n,m+1/2} = h^{-1} + \mathcal{O}(1)$, $\operatorname{Re} z_{n,m+1/2} = -\operatorname{Im}\widetilde{\zeta}_{n,m+1/2}$. These even-parity eigenvalues prove Proposition B.1, and one can take for C_0 any value greater than $\frac{(\pi/2)^2}{\sigma^3(2\widetilde{B})^{1/2}}$. □

We remark that the leading order of $k_{m+1/2}$ corresponds to the even spectrum of the operator $-h^2\partial^2/\partial_x^2$ on the undamped interval $[-\sigma,\sigma]$, with Dirichlet boundary conditions. The eigenmode $v_{n,m+1/2}$ associated with $\widetilde{\zeta}_{n,m+1/2}$ is indeed essentially supported on that interval, where it resembles the Dirichlet eigenmode $\cos\big(x\pi(1/2+m)/\sigma\big)$. At the boundary of that interval, it takes the value

$$v_{n,m+1/2}(\sigma) = (-1)^{m+1}e^{i3\pi/4}\,h^{1/2}\frac{\pi(m+1/2)}{\sigma\widetilde{B}^{1/2}} + \mathcal{O}(h)\,,$$

and decays exponentially fast inside the damping region, with a "penetration length" $(\operatorname{Im} k')^{-1} \approx (2h/\widetilde{B})^{1/2}$. From (B.3) we see that the intensity $|v_{n,m+1/2}(\sigma)|^2 \sim C\,h$ penetrating on a distance $\sim h^{1/2}$ exactly accounts for the size $\sim h^{3/2} = h\,h^{1/2}$ of the $\operatorname{Re} z_{n,m+1/2}$.

We notice that the smallest damping occurs for the state $v_{n,1/2}$ resembling the ground state of the Dirichlet Laplacian.

[1] Recall that we only need to study values $\operatorname{Re} k \ge 0$.

Odd eigenmodes

For completeness we also investigate the odd-parity eigenmodes with $k = \mathcal{O}(1)$. The computations are very similar as in the even-parity case. The odd quantization condition reads in this régime

$$\tan(k\sigma) = i\frac{k}{k'}\left(1 + \mathcal{O}(e^{-(2B/h)^{1/2}})\right). \tag{B.14}$$

The r.h.s. is then very small, showing that σk is close to a zero of the tangent, so we may take $k_m = \pi m/\sigma + \delta k_m$ with $|\delta k_m| \ll 1$ and $0 \le m \le M$. We easily see that the case $m = 0$ does not lead to a solution. For the case $m > 0$ we get

$$\delta k_m = e^{3i\pi/4}\, h^{1/2}\, \frac{\pi m}{\sigma^2 B^{1/2}} + \mathcal{O}(h)\,,$$

and thus

$$k_m = \frac{\pi m}{\sigma}\Big(1 + h^{1/2}\,\frac{e^{3i\pi/4}}{\sigma B^{1/2}} + \mathcal{O}(h)\Big), \quad 1 \le m \le M\,.$$

These values k_m approximately sit on the same "line" $\{s(1 + h^{1/2}\,\frac{e^{3i\pi/4}}{\sigma B^{1/2}}),\ s \in \mathbb{R}\}$ as the values $k_{m+1/2}$ corresponding to the even eigenmodes, both types of eigenvalues appearing successively. The corresponding energy parameter $\widetilde{\zeta}_{n,m}$ satisfies

$$\operatorname{Im}\widetilde{\zeta}_{n,m} = h^{3/2}\,\frac{(\pi m)^2}{\sigma^3(2\widetilde{B})^{1/2}} + \mathcal{O}(h^2)\,. \tag{B.15}$$

As in the even parity case, the eigenmodes $v_{n,m}$ are close to the odd eigenmodes $\sin\big(x\pi m/\sigma\big)$ of the semiclassical Dirichlet Laplacian on $[-\sigma, \sigma]$, and penetrate on a length $\sim h^{1/2}$ inside the damped region.

The case of the square

If the torus is replaced by the square $[-1/2, 1/2] \times [0, 1]$ with Dirichlet boundary conditions, with the same damping function (B.1), the eigenmodes $P(z)$ can as well be factorized into $u(x, y) = \sin(2\pi ny)v(x)$, with $n \in \frac{1}{2}\mathbb{N} \setminus 0$, and $v(x)$ must be an eigenmode of the operator (B.6) vanishing at $x = \pm 1/2$. We notice that the odd-parity eigenstates (B) satisfy this boundary conditions, so the eigenvalues $z_{n,m}$ (with real parts given by (B.15)) belong to the spectrum of the damped Dirichlet problem.

Similarly, in the case of Neumann boundary conditions the eigenmodes factorize as $u(x, y) = \cos(2\pi ny)v(x)$, with $n \in \frac{1}{2}\mathbb{N}$. The even-parity states (B.8) satisfy the Neumann boundary conditions at $x = \pm 1/2$, so that the eigenvalues $z_{n,m+1/2}$ described in (B.13) belong to the Neumann spectrum.

As a result, the Dirichlet and Neumann spectra also satisfy Proposition B.1.

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